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Axial vectors, skew-symmetric tensors and the nature of the magnetic field

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Abstract

The direction assigned to the magnetic field today derives historically from magnetic navigation. The author argues that this is a null direction and that the true action of the magnetic field occurs in a plane perpendicular to the conventional direction. This has been recognized in physics since Ampère, but the momentum of tradition and the mathematical convenience of established conventions and notation appears to have prevented its widespread acceptance.

1. The mathematical structure of the magnetic field: vector or tensor?

The direction of the magnetic field today remains that which William Gilbert (1544–1603) and magnetic navigation have assigned to it, and the field is still widely thought to possess a positive power of orienting a compass needle in this direction [1]. The discovery of electromagnetism in 1820, and the gradual shift of attention to the magnetic forces produced by, and acting on, electric currents, introduced certain ambiguities to the concept of the direction of the magnetic field. André Marie Ampère (1775–1836), in his seminal study of electrodynamics in 1823, found that a plane of electrodynamic action could be identified at any point in the neighbourhood of a closed current-bearing circuit. When a current element lies in this plane it experiences a force directed along the plane and perpendicular to the element, whatever the orientation of the latter. This plane was perpendicular to the conventional direction of the magnetic intensity. Ampère called it the ‘directive plane’ [2]. He also introduced a quantity perpendicular to this plane, which has the direction of the conventional intensity, and which he called the ‘directrix’. The magnetic intensity in the nineteenth century was generally represented by Ampere’s ‘directrix’ function, rather than by his ‘directive plane’.

Mathematical investigations in the latter part of the 19th century began to reveal that the magnetic intensity has curious structural features. Hermann von Helmholtz (1821–1894) in 1858 drew attention to the close mathematical analogy between vortex motion and the magnetic intensity of an electric current [3]. Emil Wiechert (1861–1928) in 1899 may have been the first to recognize that Ampère’s directrix, or magnetic intensity vector, remains unaltered under a coordinate inversion and, therefore, has the mathematical properties of a rotation. He describes it as a rotation vector or ‘rotor’ [4]. Arnold Sommerfeld (1868–1951) in 1910 distinguished between the terms ‘polar’ vector and ‘axial’ vector in electromagnetism and states that the magnetic intensity is an axial vector [5].

Ampère's 'directive plane' began to receive more mathematical attention in the early twentieth century. The bringing together of axial vector theory, anti-symmetric tensor theory and planar dyadic theory [6] led Sommerfeld in 1910 to the recognition that the magnetic intensity is not a vector but an anti-symmetric tensor of second rank, and that it is a '... planar magnitude (plangrössen)'. The plane in which this tensor acts is, of course, Ampère's 'directive plane'. Sommerfeld states that he will retain the customary notation H_x, H_y and H_z for the magnetic intensity, but that H_{yz}, H_{zx} and H_{xy} would be more correct [7]. Hermann Weyl (1885–1955), in 1918, criticized the axial vector representation of the magnetic field quite strongly [8]:

> 'It may be justifiable on the grounds of economy of expression to replace skew-symmetrical tensors by vectors in ordinary vector analysis, but in some ways it hides the essential feature; it gives rise to the well-known "swimming rules" in electrodynamics, which in no wise signify that there is a unique direction of twist in the space in which electrodynamic events occur; they become necessary only because the magnetic intensity of field is regarded as a vector, whereas it is, in reality, a skew-symmetric tensor (like the so-called vectorial product of two vectors). If we had been given one more space-dimension, this could never have occurred.'

During the same period Albert Einstein (1879–1955) made similar criticisms of the axial vector representation of the magnetic intensity [9]. Although Einstein went on to sketch a theory of three-dimensional electromagnetism in index notation, in which the electric field is treated as a polar vector and the magnetic field as an anti-symmetric tensor, this approach has never become widespread [10]. Elementary presentations continue to find it convenient to treat the magnetic field as an axial vector [11], while advanced treatments combine the electric and magnetic fields into a single anti-symmetric four-dimensional tensor. Sommerfeld's magnetic tensor does appear there—as a component of the four-tensor [12].

The systematic development of the approach initiated by Sommerfeld and Einstein throws interesting light on the nature of magnetism. It also distinguishes sharply between the structure of the electric and magnetic fields, a distinction that is sometimes overlooked.

2. The plane of action of the magnetic field

What are the grounds for maintaining that the magnetic field is more appropriately represented by an anti-symmetric tensor rather than by a vector? Consider Maxwell's differential version of the law of Faraday and Neumann:

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}. \tag{1}$$

The electric field strength $\boldsymbol{E}$, which changes sign under an inversion of coordinate axes, is termed a *polar* vector. However, the components of $\nabla \times \boldsymbol{E}$ do not change sign under such an inversion, and it is termed an *axial* vector. It follows that the components of $\boldsymbol{B}$ do not change sign either, under a coordinate inversion, revealing that $\boldsymbol{B}$, also, is an axial vector.

The curl of $\boldsymbol{E}$ identifies a structure in the electric field which is equivalent to a vortex or circulation in the field, and which defines a plane perpendicular to the conventional direction of $\nabla \times \boldsymbol{E}$[1]. It is more accurate to represent such a vortex by a planar antisymmetric tensor than by a vector. This can be demonstrated as follows. If its planar structure is made explicit by writing $|\nabla \times \boldsymbol{E}|_{xy}$ instead of $|\nabla \times \boldsymbol{E}|_z$ and running the suffixes through all possibilities, then

$$|\nabla \times \boldsymbol{E}|_{\alpha\beta} = \frac{\partial E_\beta}{\partial x_\alpha} - \frac{\partial E_\alpha}{\partial x_\beta}. \tag{2}$$

[1] If $\boldsymbol{n}\Delta s$ represents a circular area vector with its normal $\boldsymbol{n}$ parallel to $\nabla \times \boldsymbol{E}$, then $\int_{\Delta s} \nabla \times \boldsymbol{E}.\ \boldsymbol{n}\,\mathrm{d}s = \int_c \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l}$, from Stokes' theorem, where $\mathrm{d}s$ is an element of the area Δs and $\mathrm{d}\boldsymbol{l}$ is a vector element of its periphery. If this is non-zero it means that $\boldsymbol{E}$ has a mean component around the periphery of the loop and has, therefore, a vorticity in the plane of Δs. For a rigorous treatment see [13].

Nine sets of values result, three of which are zero and three of which are the negatives of the remaining three. It can easily be shown that this quantity transforms as a second-rank tensor [14]. Although it contains more terms than $\nabla \times \boldsymbol{E}$, it is a better physical description of the vorticity because it directly represents its plane and sense of rotation. For an axial vector, the sense of circulation is a pure convention that has to be read into the notation. For an antisymmetric planar tensor, however, the sense of circulation is a straightforward interpretation of its alternative signs. Equation (1) implies that B also is best represented mathematically as a planar tensor:

$$B_{\alpha\beta} = \begin{vmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{vmatrix} = \begin{vmatrix} 0 & B_z & -B_y \\ -B_z & 0 & B_x \\ B_y & -B_x & 0 \end{vmatrix} = \begin{vmatrix} 0 & B_{xy} & -B_{zx} \\ -B_{yz} & 0 & B_{yz} \\ B_{zx} & -B_{yz} & 0 \end{vmatrix} \tag{3}$$

where $\alpha = 1, 2, 3$ and $\beta = 1, 2, 3$. $B_{\alpha\beta}$ is obviously anti-symmetric since $B_{\alpha\beta} = -B_{\beta\alpha}$.

If $\boldsymbol{B}$ is properly a tensor why is it possible to represent it as an axial vector? In three-dimensional space an axial vector happens to represent the same geometrical information as a skew-symmetric tensor, since both define an axis of rotation, a corresponding family of parallel planes, an orientation in each plane and are isomorphic to one another [15][2]. In higher or lower dimensional spaces this isomorphism breaks down[3]. In a four-dimensional space, for example, $|\nabla \times \boldsymbol{E}|_{\alpha\beta}$ would have six independent components while an axial vector would only have four, and could not represent the hypervorticity. The possibility of representing the vorticity of the electric field—and the intensity of the magnetic field—by a vector is, therefore, an accident of three-dimensional space.

It is well known, of course, that other quantities in physics that are commonly represented by axial vectors, such as torque, angular velocity and angular momentum, represent actions taking place around an axis, not along the axis. Indeed, they represent physical processes taking place in planes perpendicular to the axis and are more appropriately represented physically by antisymmetric tensors (Feynman *et al* [11, vol 1] and [16]). Does the magnetic field also represent a physical processes taking place in a plane perpendicular to an axis, and does it have a sense of twist in that plane? For further illumination let us now turn to experiment.

As is well known, experiment reveals that that the direction of the force exerted by the magnetic field on a moving charge q is *perpendicular* to $\boldsymbol{B}$. Clearly, the direction assigned to the magnetic field is not the direction in which the field acts on a moving charge. By contrast, the direction given to the electric field is, of course, the direction of action of that field. All of this is expressed by the Lorentz force law, $\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$. Again, when a magnetic field acts on a current element, whatever the inclination or orientation of the element at a given point, the force invariably acts in a locally fixed plane—Ampère's directive plane—which is perpendicular to the conventional direction of the magnetic field (figure 1). When a current element lies in this plane it experiences a force along the plane and perpendicular to the element, which, in scalar magnitude, is independent of the orientation of the element. At any point, therefore, *the propensity of the magnetic field to act does not have a fixed direction, it has a fixed plane of action* (somewhat like surface tension) and it can act equally in any direction in its plane. For that reason, if it acts in the xy plane only, for example, it is most conveniently represented, in scalar notation, by B_{xy}.

The magnetic field also has an orientation in its plane of action. A free charge moving in the local plane of action will orbit clockwise or anticlockwise, depending on the local sense of **B** and the sign of the charge (figure 2). If a positive charge orbits *clockwise*, the sense of the field in relation to the observer will be conventionally called *positive*. This is not the usual convention but it is demanded by the left-handedness of Fleming's rule, and by the adoption of the right-handed grip or screw rule to define positive rotations. The fact that **B** has a sense of rotation or twist is also revealed by the change in sign of **B** under time reversal.

[2] For a proof of isomorphism see footnote 6 later.

[3] In an n-dimensional space antisymmetric tensors will have $(n^2 - n)/2$ independent components since there will be n^2 terms, less n zero-valued diagonal terms, and each of the remaining terms appears twice—with opposite signs. Vectors, however, will have only n independent components, and $n = (n^2 - n)/2$ only when $n = 3$.

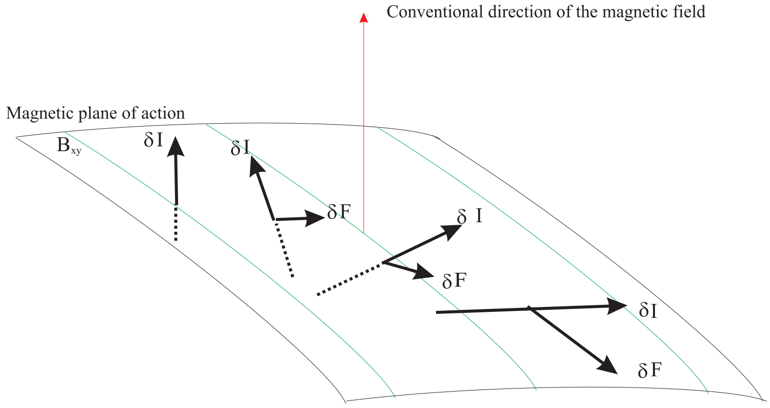

Figure 1. The force on a current element that is inclined to the directive plane of the magnetic field. The force always lies in the plane of action of the field. When the element is perpendicular to the field the force is zero. The contours represent lines of equal magnetic field strength.

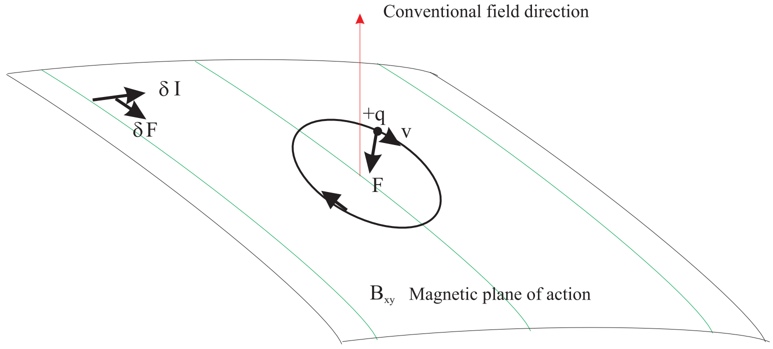

Figure 2. The path of a charged particle in a magnetic field. The path is a clockwise circle if the particle is positive and moving in the plane of action of a uniform positive field. It will be a helix if the velocity is inclined to this plane. The field is positive when the magnetic force $\delta\boldsymbol{F}$ bears 90° clockwise from the current element $\delta\boldsymbol{I}$.

After much consideration I have decided to represent or key the sense of the magnetic plane by the direction of the force on a current element, with a given orientation, lying in the plane of action of the field. This can easily be extended to any other orientation and the key can readily be slid in thought across the plane of action. The sense of the magnetic field is positive if a clockwise rotation of the current element through a right angle in the plane of the field gives the direction of the force acting upon it.

When a magnetic needle lies with its axis in the plane of action it experiences maximum torque, with the magnitude of the torque independent of its orientation in the plane. This torque reduces to zero when the needle axis is perpendicular to the magnetic plane of force. Clearly, the magnetic field moves, or tends to move, a magnetic needle *out* of its plane of action. Therefore, the conventional direction of the magnetic field is a null direction of magnetic force and represents the normal to the plane of action of the field. The magnetic field consists, therefore, not of lines of force but of a family of planes of force perpendicular to these lines (figure 3). The tensor representation of the magnetic field models all of these actions explicitly[4].

[4] For a permanent magnet, the external magnetic surfaces are, of course, identical with the external scalar equipotential surfaces, since both sets of surfaces are perpendicular to the conventional field direction.

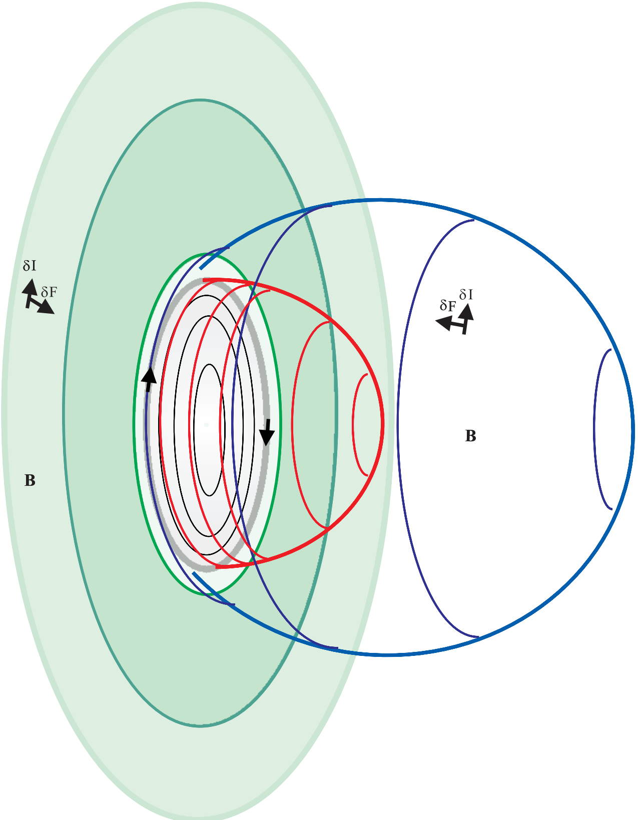

Figure 3. Magnetic surfaces surrounding a circular current-bearing coil. The surfaces are everywhere perpendicular to the conventional magnetic field direction. They are similar to the surfaces of the Earth's magnetic field. The keys give the sense of the field. They can be moved around a surface and rotated to give the sense in other positions.

3. Choosing a representation for the magnetic field tensor

Tensors in physics are usually defined rather abstractly in terms of their transformational properties rather than in terms of their physical properties. It could be said that many tensors, like vectors, are directed quantities, but, unlike vectors, cannot be defined in terms of a single direction: they are multi-directional in some sense, and there is a great variety of such senses. Surface tension is a tensor of this kind: within its surface it can act in any direction, the direction in any concrete case being determined by the edge offered to the liquid surface. It always acts in its local plane and in a direction perpendicular to any edge in that plane, just like the magnetic field tensor.

The magnetic field is a second-rank anti-symmetric tensor acting in planes in three-dimensional Cartesian space. Rather than index notation I have chosen bivectors[5] to represent it because they express particularly clearly the plane of action of the magnetic field, and make many of the basic equations of electromagnetism easier to interpret. The calculus of exterior forms and Clifford algebras may also, of course, be used in more advanced representations of electromagnetism (Frankel [11] and [17]). I shall use SI units throughout this article.

[5] I would like to acknowledge my debt to Professor Eoghan McKenna, of fond memory, of University College Galway, who introduced me to tensor calculus using vector notation in the late 1950s.

The magnetic field strength B may be represented using tensor products of unit vectors, as follows [18][6]:

$$\mathsf{B} = B_{xy}(\boldsymbol{ij} - \boldsymbol{ji}) + B_{yz}(\boldsymbol{jk} - \boldsymbol{kj}) + B_{zx}(\boldsymbol{ki} - \boldsymbol{ik}). \tag{4}$$

Here B, the scalar magnitude of B, is the magnitude of the force acting in the plane of B and perpendicular to a unit current element placed with any orientation in that plane. B_{xy}, B_{yz} and B_{zx} represent the scalar components of B in the reference planes. From axial vector theory

$$B = (B_{xy}^2 + B_{yz}^2 + B_{zx}^2)^{1/2}. \tag{5}$$

The direction cosines between the normal to B and the coordinate axes are given by

$$l_x = B_{yz}/B \qquad l_y = B_{zx}/B \qquad l_z = B_{xy}/B.$$

The axial vector equivalent to B is [7]

$$\boldsymbol{B} = \boldsymbol{i}B_{yz} + \boldsymbol{j}B_{zx} + \boldsymbol{k}B_{xy} \qquad \text{or} \qquad \boldsymbol{B} = \boldsymbol{i}B_x + \boldsymbol{j}B_y + \boldsymbol{k}B_z. \tag{6}$$

Of course, just as torque can be represented by an axial vector and yet thought of as a process lying in a plane, so too the magnetic field can be *thought of* as a process lying in a plane and yet be *represented* by an axial vector, or even by a scalar. A particular physical interpretation need not be tied to a particular mathematical representation.

The dot or inner product of B with a vector (as a postfactor) rotates the component of the vector in the plane of B clockwise through 90°, if B is positive. The component of the vector perpendicular to the plane of B is eliminated. For example,

$$\mathsf{B} \cdot \boldsymbol{B} = 0$$

confirming that the plane of the magnetic tensor is perpendicular to the conventional direction of the field.

It is now common to write alternating or 'outer' products of two vectors as

$$\boldsymbol{ab} - \boldsymbol{ba} = \boldsymbol{a} \wedge \boldsymbol{b}. \tag{7}$$

This is sometimes called the 'wedge' product of $\boldsymbol{a}$ and $\boldsymbol{b}$ [19]. The 'wedge' symbol $\wedge$ should *not* be confused with the symbol for the vector product, although here it has the same magnitude as the latter—since $|\boldsymbol{a} \wedge \boldsymbol{b}| = |\boldsymbol{ab} - \boldsymbol{ba}| = ab \sin \angle ab$, where $\angle ab$ is the acute angle between $\boldsymbol{a}$ and $\boldsymbol{b}$. The plane of $\boldsymbol{a} \wedge \boldsymbol{b}$ (or of $\boldsymbol{ab} - \boldsymbol{ba}$) is, of course, perpendicular to the axis of $\boldsymbol{a} \times \boldsymbol{b}$. Unless $\boldsymbol{a} \wedge \boldsymbol{b}$ is interpreted as a property in the plane specified by the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ it may seem a piece of abstract formalism. In this notation

$$\mathsf{B} = B_{xy}(\boldsymbol{i} \wedge \boldsymbol{j}) + B_{yz}(\boldsymbol{j} \wedge \boldsymbol{k}) + B_{zx}(\boldsymbol{k} \wedge \boldsymbol{i}). \tag{8}$$

Each of the unit bivectors $(\boldsymbol{i} \wedge \boldsymbol{j})$, $(\boldsymbol{j} \wedge \boldsymbol{k})$ and $(\boldsymbol{k} \wedge \boldsymbol{i})$ represents a plane, and a unit rotational operator in that plane, since its dot product with a coplanar vector rotates the latter in this plane clockwise, without change of scale or dimensionality, through 90°. Here unit bivectors have an analogous role to that of unit vectors in polar vector theory.

[6] In formal mathematical terms the tensor product of two vectors is a bilinear, real valued function defined on vectors such that $ij\ (a, b) = (i * a)(j * b)$ where '$*$' is the ordinary dot product. The contraction of a bivector with a vector is just the vector obtained when the first slot of the bilinear function is used: $ij(a,\ ) = (i * a)j$.

[7] B is clearly invariant under an inversion of axes, since replacing $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ by $-\boldsymbol{i}$, $-\boldsymbol{j}$, $-\boldsymbol{k}$ does not alter B. It is plain also that B is antisymmetric since its conjugate $\mathsf{B}_c = B_{xy}(\boldsymbol{ji} - \boldsymbol{ij}) + B_{yz}(\boldsymbol{kj} - \boldsymbol{jk}) + B_{zx}(\boldsymbol{ik} - \boldsymbol{ki}) = -\mathsf{B}$. We can transform from B to B and *vice versa* using the relations $\mathsf{B} = \mathsf{E} \cdot B$ and $B = \frac{1}{2}\mathsf{B}{:}\mathsf{E}$, where $\mathsf{E} = (\boldsymbol{ijk} + \boldsymbol{jki} + \boldsymbol{kij} - \boldsymbol{ikj} - \boldsymbol{kji} - \boldsymbol{jik}) = \boldsymbol{i} \wedge \boldsymbol{j} \wedge \boldsymbol{k}$. In index notation, $B_{jk} = e_{jkl}B_l$ and $B_i = \frac{1}{2}e_{ijk}B_{jk}$, where $e_{ijk} = \pm 1$ if i, j, k are even or odd permutations of 1, 2, 3, respectively, and $e_{ijk} = 0$ if any two of i, j, k are the same. I am grateful to Geoff Brooker for a very helpful discussion of the relationships between index and dyadic notation in magnetism.

The outer curl or *vorticity* of the polar vector $\boldsymbol{E}$ is the planar tensor

$$\nabla \wedge \boldsymbol{E} = \nabla \boldsymbol{E} - (\nabla \boldsymbol{E})_{\mathrm{c}} = (\boldsymbol{i} \wedge \boldsymbol{j}) \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) + (\boldsymbol{j} \wedge \boldsymbol{k}) \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right)$$
$$+ (\boldsymbol{k} \wedge \boldsymbol{i}) \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \right). \tag{9}$$

If $\boldsymbol{A}$ represents the vector potential then

$$\mathbf{B} = \nabla \wedge \boldsymbol{A}. \tag{10}$$

4. Applications to electrodynamics

The basic relations of electrodynamics are represented in this notation as follows[8].

The force on a current element $I\delta\boldsymbol{l}$ is related to the tensor field $\mathbf{B}$ by

$$\boldsymbol{F} = I\mathbf{B} \cdot \delta\boldsymbol{l}. \tag{11}$$

This indicates that the force $\boldsymbol{F}$ is in the plane of $\mathbf{B}$ and perpendicular to $\delta\boldsymbol{l}$ (figure 1). Similarly (figure 2), the force on a moving charge in a magnetic field is expressed by

$$\boldsymbol{F} = q\mathbf{B} \cdot \boldsymbol{v}. \tag{12}$$

Also, the Lorentz force law on a charge q in a combined electric and magnetic field becomes

$$\boldsymbol{F} = q(\boldsymbol{E} + \mathbf{B} \cdot \boldsymbol{v}). \tag{13}$$

The magnetic field strength at P is related to its source element $I_1\delta l_1$ (figure 4) by

$$\delta\mathbf{B} = \frac{\mu_0}{4\pi} I_1 \left(\frac{\delta\boldsymbol{l}_1\boldsymbol{r} - \boldsymbol{r}\delta\boldsymbol{l}_1}{r^3} \right) = \frac{\mu_0}{4\pi} I_1 \left(\frac{\delta\boldsymbol{l}_1 \wedge \boldsymbol{r}}{r^3} \right) \tag{14}$$

where r is the position vector of P with respect to the element. Similarly, the force on a second current element $I_2\delta l_2$ at P is

$$\delta\boldsymbol{F} = \delta\mathbf{B} \cdot (I_2\delta\boldsymbol{l}_2) = \frac{\mu_0}{4\pi} I_1 I_2 \left(\frac{\delta\boldsymbol{l}_1\boldsymbol{r} - \boldsymbol{r}\delta\boldsymbol{l}_1}{r^3} \right) \cdot \delta\boldsymbol{l}_2 = \frac{\mu_0}{4\pi} I_1 I_2 \left(\frac{\delta\boldsymbol{l}_1 \wedge \boldsymbol{r}}{r^3} \right) \cdot \delta\boldsymbol{l}_2. \tag{15}$$

I find this more intuitive physically than

$$\delta\boldsymbol{F} = \frac{\mu_0}{4\pi} I_1 I_2 \delta\boldsymbol{l}_2 \times \left(\frac{\delta\boldsymbol{l}_1 \times \boldsymbol{r}}{r^3} \right)$$

because the former expressly reveals that the force on the second current element lies in the plane defined by the first current element, and $\boldsymbol{r}$, and is perpendicular to $\delta\boldsymbol{l}_2$. I have always found the cross-product expression here difficult to visualize.

Maxwell's magnetic vorticity equation becomes a divergence equation in this notation. Indeed, $\nabla \times \boldsymbol{B}$ is a polar vector and not a vorticity:

$$\nabla \cdot \mathbf{B} = \mu_0 \boldsymbol{j} + \frac{1}{c^2} \frac{\partial \boldsymbol{E}}{\partial t}. \tag{16}$$

When $\partial\boldsymbol{E}/\partial t = 0$,

$$\nabla \cdot \mathbf{B} = \mu_0 \boldsymbol{j}.$$

$\nabla \cdot \mathbf{B}$ represents a true divergence of the planar magnetic field, as can be seen from figure 5. This shows that an electric current gives rise to a divergence of the magnetic field, just as an

[8] Some of the relations that follow are already well-known in other notations: see Einstein A 1967 *The Meaning of Relativity* (London: Methuen) p 21–2, 38–40 (1922 1st German edn) and Frankel T 1997 *The Geometry of Physics: an Introduction* (Cambridge: Cambridge University Press) p 119. Here they are partly based on the identity $\boldsymbol{a} \times \boldsymbol{T} = \mathbf{T} \cdot \boldsymbol{a}$, for any vector $\boldsymbol{a}$, where $\boldsymbol{T}$ is the axial vector corresponding to the anti-symmetric tensor $\mathbf{T}$.

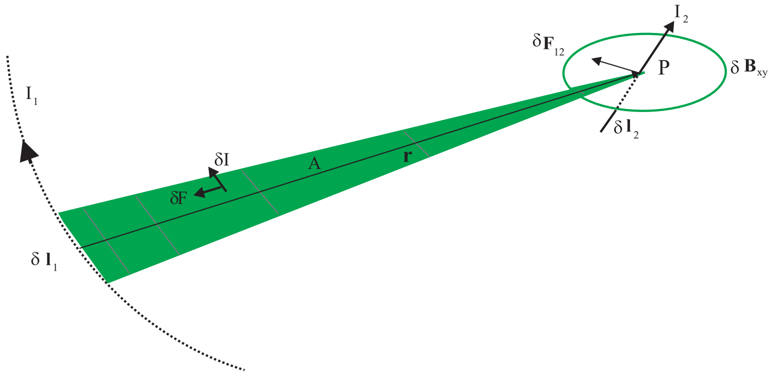

Figure 4. The magnetic plane of force of a current element. Note that $\delta \boldsymbol{l}_1$, $\boldsymbol{r}$, $\delta\mathbf{B}$ and $\delta \boldsymbol{F}_{12}$ all lie in the same plane.

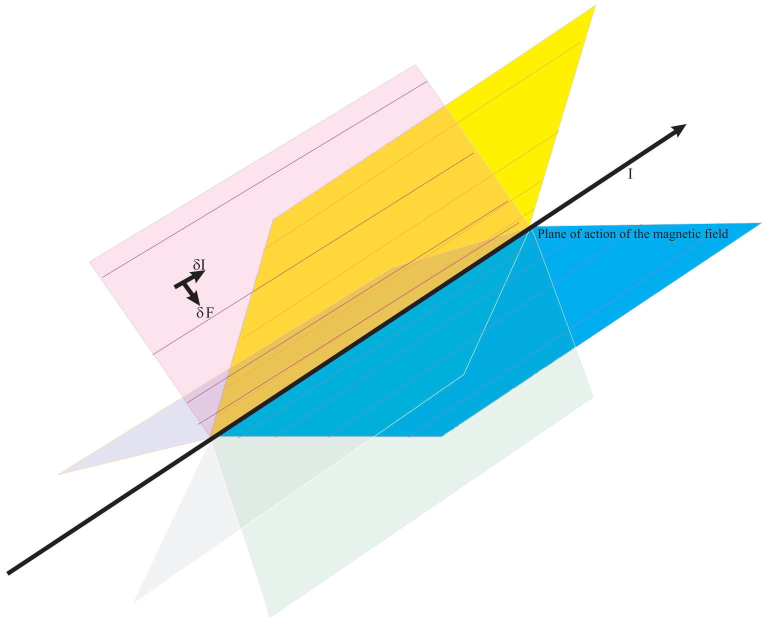

Figure 5. Magnetic planes of force diverging from a long straight current-bearing conductor.

electric charge gives rise to a divergence of the electric field. Since the normals to the magnetic field never diverge ($\nabla \cdot \boldsymbol{B} = 0$, always)

$$\frac{\partial B_{yz}}{\partial x} + \frac{\partial B_{zx}}{\partial y} + \frac{\partial B_{xy}}{\partial z} = 0$$

or, in full,

$$\frac{\partial B_{\alpha\beta}}{\partial x_\gamma} + \frac{\partial B_{\beta\gamma}}{\partial x_\alpha} + \frac{\partial B_{\gamma\alpha}}{\partial x_\beta} = 0. \tag{17}$$

This means that the surfaces of action of the magnetic field never close—unlike the lines of action of the electric field.

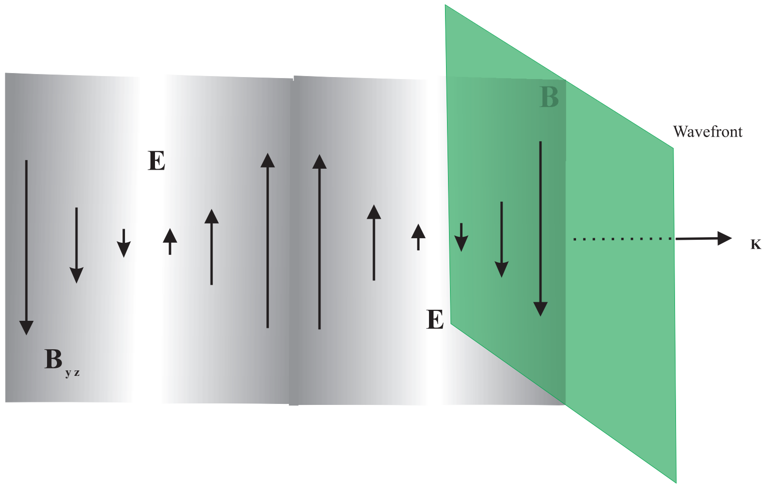

Figure 6. A plane sinusoidal radiation field in free space. Note that $\boldsymbol{E}$ and $\mathbf{B}$ lie in the same plane, the plane of polarization. The vector potential also lies in that plane.

From this perspective, what is commonly termed the *circulation* of the magnetic field becomes a measure of its divergence and might be termed the *sweep* of the field around the current. Similarly, the *flux* φ of the field is a flux of the normals to the field. Without suggesting that a change of nomenclature is required, a more accurate descriptive term for φ here might be the *cover* of the field.

In the present notation Maxwell's electric vorticity equation takes the form

$$\nabla \wedge \boldsymbol{E} = -\frac{\partial \mathbf{B}}{\partial t}. \tag{18}$$

This analysis suggests that Maxwell's time-dependent equations are structurally dissimilar, in that the first is a polar vector equation and the second a bivector equation. Furthermore, (17) is a third-rank tensor equation. Maxwell's equations therefore lose much of their comparative symmetry in this notation, suggesting that the degree of symmetry present in the axial vector representation of these equations was only apparent.

The electric and magnetic radiation field strengths are related by the expression

$$\mathbf{B} = \frac{1}{c}(\boldsymbol{n}\boldsymbol{E} - \boldsymbol{E}\boldsymbol{n}) = \frac{1}{c}(\boldsymbol{n} \wedge \boldsymbol{E}) \tag{19}$$

where $\boldsymbol{n}$ is the unit normal to the wavefront. This reveals that $\boldsymbol{E}$ and $\mathbf{B}$ lie in the same plane, which, for a plane wave, is the plane of polarization (figure 6). Also,

$$\nabla^2 \mathbf{B} = \frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2} \tag{20}$$

which shows that $\mathbf{B}$ travels as a tensor wave.

5. Magnetic 'swimming rules'

It is possible to see certain pedagogical advantages in this perspective on the magnetic field. What Weyl amusingly terms the 'swimming rules' (Fleming's left- and right-hand rules) would no longer be needed. Also, when one becomes thoroughly familiar with this approach, certain phenomena become easier to visualize. Figure 7 represents the magnetic forces on a circular current bearing coil placed in the magnetic field near one end of a bar magnet. By simply

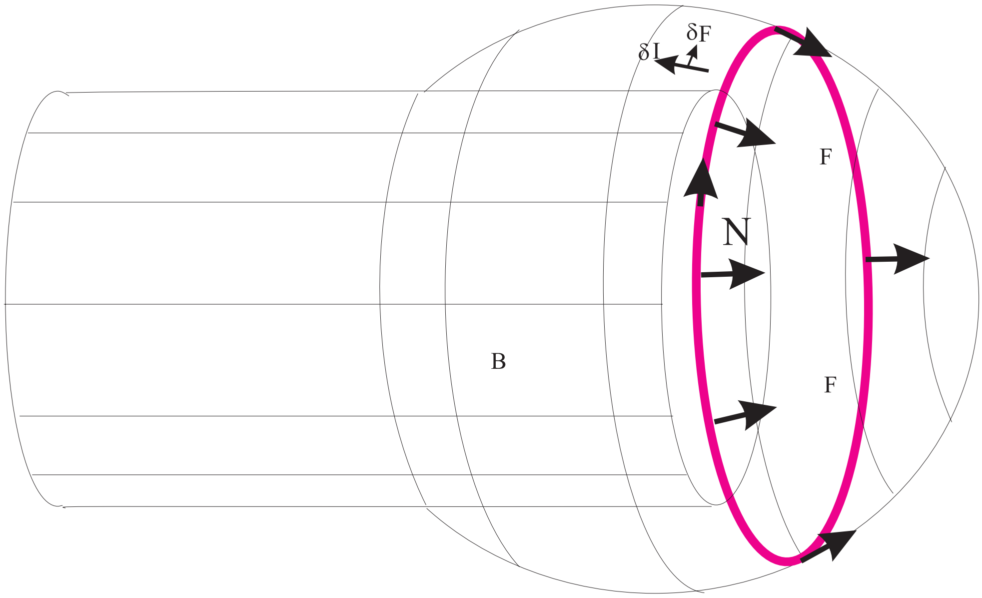

Figure 7. Forces on a circular current-bearing coil in a curved magnetic field due to a bar magnet. The resultant force on the coil will be along its axis.

inspecting the diagram we can immediately tell the line of action of the force on the coil. The key provided in the diagram gives the sense of this force.

If the objection is made that the direction of the magnetic field is much easier to visualize in the conventional representation, it is important to consider whether this apparent simplicity is bought at too high a price: do we not pay for it in the complexity of the finger and wrist manoeuvres required to translate it into force directions on currents, and into the field directions of currents? And do we find it much easier only because we are more used to it?

The same figure can be used to illustrate motional electromagnetic induction in the vicinity of the magnet. Suppose a vertical wire moves transversely to its own length, and parallel to the local surface of magnetic action. The field will interpret this motion as the horizontal convection of conduction charges along its surface. It will exert a magnetic force perpendicular to this motion, along the wire, and this acts as a Lorentz electromotive force in the wire. Using the key we can find the direction of this force. An advantage of this approach is that it reduces the problem of determining directions from three dimensions to two.

In elementary approaches throughout physics, vector products of polar vectors are generally treated as axial vectors rather than anti-symmetric tensors because of the mathematical convenience of doing so. This practice tends to obscure the planar character of these products. I believe that the plane of action of the magnetic field should be pointed out to physics undergraduates. Very few will know, intuitively, that the conventional direction of the field is the direction of its axis, only, because its axial nature is more hidden, and is not often represented as a vector product.

I believe that much remains to be learned about the implications of this approach. These are not simply a matter of mathematics, primarily they relate to physical interpretation.

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