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What is momentum?

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Abstract

Momentum is commonly defined as 'mass times velocity'. However, this cannot be a general definition, since it does not include other types of momentum, including macroscopic radiation momentum or the momentum of a photon. Rankine coined a general definition for energy. Is it possible to describe a corresponding definition of momentum?

> 'There is a part of everything which is unexplored . . . Even in the smallest thing there is something in it which is unknown. We must find it'. Gustav Flaubert (1821–1880) quoted by Guy de Maupassant (1850–1893) [144].

1. Introduction

Today, the term 'momentum' is universally recognized as an authentic and useful concept, and its common definition is 'mass times velocity'. In certain cases, however, this definition does not apply. It does not apply to relativistic mechanical momentum ($\boldsymbol{p} = \gamma m\boldsymbol{v}$), to radiation momentum $\left(\boldsymbol{g} = \frac{1}{c^2\mu_o}\boldsymbol{B}\cdot\boldsymbol{E}\right)$, to the effective momentum of a stationary charge in the neighbourhood of a current or magnet ($\boldsymbol{p} = q\boldsymbol{A}$), nor to the photon momentum ($\boldsymbol{p} = h\boldsymbol{k}$). In fact, in no case is 'mass times velocity' a rigorous definition of momentum.

Sometimes in physics, to move forwards, it is helpful to look backwards. Carrying out a critical analysis of all of significant primary sources of a particular topic, in chronological order, and in the hands of an experienced physicist, applies a richer resource to tackling unsolved problems. It can also give an enhanced role to neglected concepts, and it can refresh older definitions. I attempt to apply this method to the concept of momentum. It also leads to an examination of the concept of force, of Newton's second law, of the 'force of inertia' and of the concept of impulse. I believe that each of these concepts also needs renewal.

2. Impetus and impact

Since the 17th century different vernaculars have used different terms for momentum, including 'impetus', 'momentum', 'quantity of motion' and 'impulse'. This reflects uncertainty in this concept. Since the Middle Ages the mathematical definition of momentum has remained stable, but the physical interpretation of momentum has varied greatly. Because of this, it will be necessary to examine momentum in some detail.

The remote ancestor of the concept of momentum was John Philoponus of Alexandria (sixth century) [1]. Aristotle (384–322 BC), eight centuries earlier, argued that the rush of air, which accompanies a projectile at its beginning, continues to sustain it in motion [2]. Philoponus describes Aristotle's theory as 'fantastic', arguing that even if [3]

> countless machines . . . set a large quantity of air in motion behind these bodies . . . the projectile will not be moved . . . a single cubit.

Philoponus saw that the air resists, rather than sustains, the motion of an arrow or missile. So what drives a freely moving projectile? For Philoponus, 'some incorporeal motive force is imparted by the projector to the projectile'. That is, a non-material internal force now drives it. Once motion begins, therefore, 'there will be no need of any agency external to the projector' to move the projectile along. This internal force 'will be produced much more readily in a void than in a plenum' [4].

Simplicius (b. 500, d. after 533), a contemporary of Philoponus, dismissed his persisting force as an example of 'Egyptian mythology' [5]. But this did not kill the concept. In fact, Philoponus' internal force had a long and distinguished history within Islamic [6] and European physics.

In the European Middle Ages this internal force was called 'impetus'. Jean Buridan (about 1295–1358) of Paris raised impetus theory to a high level of interpretative and quantitative sophistication:

> The impetus imparted to the . . . projectile varies . . . as the velocity . . . and as the quantity of matter of the body . . . .

Buridan used 'impetus' to mean the force which first initiates movement, but also the (supposed) force which maintains its motion. As a body falls, the accumulated impetus grows. He also uses impetus to explain the constant orbital motion of the heavenly bodies [7]. Key elements of the theory of impetus survive in the modern theory of momentum.

Impetus theory deeply influenced seventeenth and eighteenth century mechanics [8]. Galileo (1564–1642), in his *Two New Sciences of* 1638, accepts a version of impetus theory close to that of Buridan [9]. Galileo, however, helps to create a more modern term by frequently making 'impetus' synonymous with 'momentum' [10]:

> . . . the impetus or momentum of the moveable . . .

However, Galileo's term 'momentum' has a meaning quite different from that intended today. Galileo's quantification of impetus was similar to that of Buridan [11], but he added a unit of impetus [12]:

> . . . the speed acquired by one-pound ball starting from rest and falling from a height of one pikestaff is always and everywhere the same, and for that reason it is well suited to stand from the impetus deriving from natural descent.

Galileo makes impetus responsible for impact [13]:

> [W]hen the pole yields to the impetus of the pile driver, four inches the first time, and two [inches] the second . . . these impacts come out unequal . . . .

He is also interested in the forces acting during the process of impact, but he does not study this in any depth [14].

Johannes Kepler (1571–1630) in 1618 first introduced the concept of 'inertia' as a spontaneous tendency in each body to come to rest [15]. For Kepler, the Earth because of its 'natural inertia ... abhors motion' [16]. Indeed, he believed that a persisting impetus, within a uniformly rotating body, is needed to maintain its motion against its natural rotational inertia [17]. Kepler's inertia made explicit what had been assumed implicitly for a thousand years in the theory of impetus.

Contrary to Kepler, Galileo argues powerfully that, in the horizontal, every body is 'indifferent to motion and rest' [18]. However, both he and most of his successors maintained a residual belief that a persisting impetus drives freely moving bodies along uniformly [19].

René Descartes (1596–1650) published his *Principia philosophia* (1644), six years after Galileo's *Two New Sciences*. Descartes, analogously with Galileo, argues that motion is relative, and that 'the transference [of] movement and rest . . . is reciprocal'. But he also writes of 'the force of a body in motion', an expression equivalent to 'impetus' [20].

Descartes discovered that, when a moving body collides with an equally free body at rest, and they continue to move together, the speed of the combined body is halved, but the product of quantity of matter and speed remains the same [21]. He generalized this product, and defined it as the 'quantity of motion'. For Descartes, God 'conserves' the total 'quantity of motion' in the Universe [22]. Reflecting on this he argued that [23]

> When a stone falls to earth, . . . if it stops and does not bounce, I think that this is because it moves the earth, and so transfers its motion to it.

'Quantity of motion' was not a force, and was effectively a kinetic concept, because matter, for Descartes, is pure extension. Nevertheless, 'quantity of motion' is not abstract speed, but matter in a state of motion [24]. In the eighteenth century it became synonymous with 'momentum'.

Although he rejected Kepler's inertia, Descartes introduced a new kind of inertia in 1639, based upon his concept of 'quantity of motion' [25]:

> . . . if two unequal bodies receive as much motion from another, this similar quantity of motion does not give so much velocity to the bigger one as to the smaller one. It can be said, in this sense, that the more matter the body contains, the more natural inertia it possesses.

As we shall see, this is equivalent to the most common meaning of inertia today—controlling inertia.

In his *Principia*, Descartes studied the impact on two free bodies in detail [26]. We now recognize that many of Descartes rules of collision are false, and the role of direction in his law of conservation of quantity of motion required clarification [27]. Others reformed these later in the century [28]. However, the creation of the concept of 'quantity of motion', the discovery of a new conservation law, the creation of a new quantitative theory of impact, the introduction of the new concept of controlling inertia, makes Descartes a central figure in this investigation.

In 1668–1669 John Wallis (1616–1703) [29], Christopher Wren (1632–1723) [30] and Christian Huygens (1629–95) [31] greatly improved the Cartesian theory of impact. They showed that 'quantity of motion' is a directed quantity, that speed and direction are equally important in calculating impact, that the directed quantity of motion always remains constant before and after impact [32], that the phenomena of impact are independent of relative motion, and that the centre of gravity remains constant during collisions. However, imperfectly elastic bodies were not well treated [33].

Wallis identifies 'quantity of motion', 'impetus' and 'momentum'. He also describes 'momentum' as 'impelling force', and as the agent of impact [34]. Wallis represented momentum algebraically for the first time,

$$V = PC$$

where V is *momentum*, P (*pondus*) is weight (in this context it is close to 'quantity of matter') and C is *celeritas* (speed) [35].

3. Force and 'the quantity of motion' (momentum) [36]

From Antiquity the primary model of force was contact force, including forces supported by or moved by machines. Originally, the concept of 'force' implied unnatural or violent action, as distinct from natural motion, and 'force' was contrasted with the 'load' acted upon. However, during the seventeenth century the concept of force became much more general [37]. Galileo first introduced the formal study of tensile forces [38]. Huygens in 1659 apparently first introduced the tension in a string as a general model and measure of force [39]. Robert Hooke (1635–1702) in 1678 first introduced a spring calibrated against weight [40].

The ancient theory of machines hardly distinguished statics and dynamics, since the machine in static equilibrium, and the machine moving uniformly, obeyed the same basic laws [41]. However, Aristotle worked out a more general theory of forces acting on uniformly moving resistive loads [42]. John Wallis, in 1670, helped to clarify the distinction between active forces and *impedimenta*, that is, non-reversible forces such as friction [43]. To this day, the study of static forces, and moving forces acting on constrained bodies, is far more important in engineering and everyday life than forces acting on free accelerating bodies.

Contact force or 'pressure' is the primary model of force for Isaac Newton (1642–1727) [44]. Newton in his *Principia* of 1687 applies his concept of 'pressure' to static forces (the pressure of a fluid) [45], to uniformly moving bodies (to moving loads) [46], and to accelerating bodies [47], but also to percussion. He attempts to apply a similar model to gravitational force [48]. He coins various special definitions of force, including 'motive force' (which accelerates bodies) and 'centripetal force'. These definitions explicitly combine description and measurement [49].

For Newton, 'pressure' or 'impressed force' is not a property, or an object, but an *action*: '. . . force is the action exerted on a body . . . ' [50]. More specifically, a force is always a dynamic interaction—of the source body on the target body.

It was an extraordinary leap of imagination for Newton to recognize that a force, acting on a freely moving body, 'does not remain in a body once the action has ceased'. For Newton, there is no persisting impetus acting on a freely moving body. He agrees with Descartes that motion and rest are reciprocal. He also states that 'every body is only with difficulty put out of its . . . state either of being at rest or moving uniformly' [51]. This is Newton's first law, known today as the law of inertia.

Even more extraordinary, given the traditional theory of mechanical forces acting against impediments, Newton recognizes that a constant 'motive force' acting on a free body continually *changes* its motion. In his second law, Newton quantifies this concept by stating that 'a change of motion is proportional to the motive force impressed . . . '[1] [52]. This remains the fundamental form of Newton's second law. The term 'motion' of course has the same mathematical meaning as 'momentum'. Contrary to Wallis, Newton never uses the term 'momentum' (implying an internal force). Instead, he substitutes the Cartesian kinematic 'quantity of motion', or simply 'motion'.

The second law, in the form 'force equals mass times acceleration' first appears in early nineteenth century [53]. Indeed, Newton does not have an explicit numerical measure or symbol for acceleration. He measures the 'acceleration of motion' as 'proportional to the velocity which it generates in a given time' [54]. This means that the derived form of

[1] From subsequent applications it is clear that Newton implies equal times in this proportion.

his law relates force, quantity of matter, velocity and time [55]. The formal verification of Newton's second law was provided by the machine invented by George Atwood (1745–1807) in 1784 [56].

The standard way of measuring force by Newton was by comparing the test force with a standard force—often the dead weight of a solid or liquid, or by means of known elastic tensions [57]. Also, having established his second law, Newton frequently used 'the change in motion' or the 'acceleration' as a proportional measure of 'motive force' [58]. Yet another proportional measure of force was through his law of gravity [59].

Having banished persisting impetus, how does a free uniformly moving body, containing no force, act as a force at the moment of impact? To deal with this, Newton argues that there is an 'inherent force' in bodies, which he calls 'the force of inertia'. '[A] body exerts this force only during a change in its state'; that is, when the state of rest, or uniform motion, is disturbed [60]. The 'force of inertia' is a temporary forward force, induced in the arresting body, which acts on the target (and not on itself). Simultaneously, the target generates an equal inertial reaction [61]. However, it took another 60 years for physics to replace persisting impetus, as the cause of impact, with Newton's 'force of inertia'.

Newton introduced the first quantitative study of inelastic collisions [62]. He combines his third law with his 'force of inertia' to prove that equal and opposite changes occur to the quantities of motion during a collision [63]. He also thereby provides the foundation of the principle of conservation of momentum. But he does not state it explicitly, because it was then strongly associated with Descartes' scalar version of that principle [64]. In fact, the explicit acceptance of the general principle conservation of momentum may have been as late as 1872 [65].

Newton invested more effort in the study of gravity than in that of any other force [66]. He established the inverse square law of force between planets, and, qualitatively, he generalized terrestrial gravity to the whole universe [67]. He identifies the total 'motive force' of gravity with 'its weight, and it may also be known from the force opposite and equal to it' [68].

Recognizing that gravity generates the same local acceleration for all bodies irrespective of mass [69], Newton introduced a new agency directly responsible for gravitational acceleration, 'accelerative force',

> a certain efficacy diffused from the centre through each of the surrounding places in order to move the bodies that are in those places.

For Newton, and for his successors, 'accelerative force' is always measured indirectly by gravitational acceleration. Also, the 'motive force' of gravity is 'the sum of actions of the accelerative force on the individual particles of the body'. This meant that 'accelerative force' is equal to the force of gravity on unit mass [70]. Newton also introduced 'accelerative force' into contact mechanics, which again meant the force density [71]. 'Accelerative force' soon became the most powerful concept in celestial dynamics. James Clerk Maxwell (1831–1879), in 1865, renamed it the 'intensity' of the 'gravitational field' [72].

Despite his immense achievements in mechanics, Newton did not include the role of work, the generation of kinetic energy, and impulse.

4. The controversy over momentum and *vis viva* (kinetic energy)

Despite the efforts by Newton to demonstrate that no persisting force is required to maintain free motion, 'the force of a moving body' remained deeply tenacious. For example, John Desaguliers (1683–1744), in 1734, writes that [73]

> [By] momentum or quantity of motion . . . I don't mean the . . . Pressure, Traction . . . but the Force which it has all the while it is moving from one place to another.

Gottfried Leibniz (1646–1716) in 1686 attempted to introduce '*living force*' (*vis viva*) in place of 'momentum', or the Cartesian 'force of a moving body'. 'Living force' (like impetus) is a force that persists, even in a free moving body [74]. It cannot be measured directly, but 'by the violent effect it can produce'—particularly by its 'power' to raise the moving body to a given height against gravity, and vice versa [75]. Also, 'living force' can be 'consumed' through the compression of a spring during its arrest [76]. Leibniz also proved that this 'power' is jointly proportional to the mass and to the square of the original speed [77]. Famously, he argued for the conservation of living force, even maintaining that, during impact, 'living force' is 'carried over into the elasticity of bodies' [78].

While criticising the Cartesians, Leibniz considerably clarified their concept of impetus and quantity of motion. He stated that 'impetus is always combined with living force'—the first statement of today's recognition that momentum and kinetic energy are always combined in a moving body [79]. He also recognized that an impressed impetus (contact force) acting on the free body, for a certain *time*, generates 'quantity of motion' [80]. However, Leibniz denied that the 'quantity of motion' is a dynamic agency. He also denied that Cartesian (or scalar) 'quantity of motion' is conserved [81].

The debate generated an enormous controversy for more than 60 years [82]. The Cartesians had their strongest case with dead impact equilibrium, in which the mass is inversely proportional to the speed. 'Living force', which was held to be responsible for such collisions, did not predict the correct impact law in this case [83].

With hindsight, the time was not yet ripe to resolve the mixture of obscure and undeveloped concepts involved [84]. Nevertheless, many important secondary issues were clarified in the 1740s. One was the clarification of the concept of force. 'The force of a moving body' was rejected, mainly by Jean D'Alembert (1717–1783) and Leonhard Euler (1707–1783), as being obscure and metaphysical [85]. They also emphasized that Newton's 'force of inertia'—a tensile contact force which occurred only during accelerations—is the agent of collisions, rather than the hypothetical persisting 'impetus' [86].

Further, d'Alembert and Euler undermined the whole debate [87] by showing that, during an arresting impact (sometimes modelled by compressing a spring [88]), a contact force F is exerted for a certain distance $\mathrm{d}s$, but *also* for a certain time $\mathrm{d}t$. Mathematically, this was represented by the simultaneous equations $\frac{1}{2}\,mv^2 = \int F\,\mathrm{d}x$ [89] and $mv = \int F\,\mathrm{d}t$ [90]. This meant that both parties were wrong (with respect to a persisting impetus) and right (with respect to the mathematics of impact). The upshot of these criticisms was that momentum (represented by mv) and *vis viva* (proportional to mv^2) effectively disappeared as physical concepts for more than a century. They did not disappear, however, as mathematical concepts. As late as 1855 the reduced status of momentum is illustrated by Arthur Morin (1813–1884) [91]:

> The products mv have received the name 'quantity of movement'; it is a convenient expression upon which one does not attach any sense other that the product of mass times velocity.

5. Lagrange and the revival of the concept of momentum

Joseph Louis Lagrange (1736–1813), in his magisterial *Mécanique analytique*, argued in 1788 that [92]

> [T]he quantity of movement of a body is the measure of that force which the body is capable of exercising against an obstacle, and which is called the *percussion*.

Lagrange closely follows Newton's analysis of inertial impact [93]. He states that

> the product of the mass and the velocity expresses the 'finite force' of a body in movement.

By 'finite force', he means—implied in his mathematics—the inertial force acting during the short time of arrest [94].

Lagrange's description of momentum, as the capability of a moving body to deliver an arresting percussion or 'impulsion' [95], is a remarkable reconciliation of Galileo's dynamic concept of momentum with Descartes' (and Newton's) more kinematic concept. It is well known by physicists to this day. Although the conventional definition of momentum remains *mass times velocity*, the modern working understanding of momentum in physics began to grow with Lagrange [96].

Jean Belanger (1790–1874) selected Lagrange's term 'impulsion' rather than 'percussion'. This is presumably because 'percussion' ambiguously included both the vector and scalar aspects of impact. He defined 'impulsion' mathematically as $\int_0^t F\,\mathrm{d}t$. For collisions, Belanger's impulsion is, of course, Newton's 'force of inertia' F acting for a certain time t [97]. Belanger, of the École Polytechique, was inspired by the great engineering physics tradition of that school, especially by the creation of the concept of 'work' by Jean Poncelet (1788–1867) and Gaspard Coriolis (1792–1843) [98]. In 1868 Maxwell combined Lagrange and Belanger's definitions [99]:

> The impulse of a force is equal to the momentum produced by it.

Even if a collision does not bring a body to rest, the change in momentum measures the impulse. Also, impulse can be quantified in principle by comparing arbitrary with standard impacts—without measuring force and time. The concept of impulse was introduced into textbooks in the late nineteenth century [100].

William Rankine (1820–1872) in 1855 combines the interpretation and measurement of the term 'energy' as follows [101]:

> The term 'energy' comprehends every state of a substance which constitutes the capacity for performing work. Quantities of energy are measured by the quantities of work which they constitute the means of performing.

William Thomson (1824–1907) in 1867 interpreted $\frac{1}{2}mv^2$ physically as 'kinetic energy'—the capacity of a moving body to perform work during arrest [102]. This meant that, after more than a century of in which $\frac{1}{2}mv^2$ was largely interpreted as an useful function, kinetic energy returned to a full physical meaning.

As a working concept in physics, nineteenth century 'momentum' also returned to physical status, but textbook definitions remained unexamined. When Rankine coined his definition of energy there was a felt need to generalize the various kinds of energy. But in his day there was only one kind of momentum—mechanical momentum. Perhaps, when new concepts of momentum eventually appeared, the great creative period of defining new dynamical concepts had passed—for the present.

New concepts of momentum did begin to emerge from the late nineteenth century. John Poynting (1852–1914) recognized radiation momentum in 1884 [103], and in 1904, James J Thomson (1856–1940) discovered that even a stationary charge has an implicit momentum near a current [104]. In 1906 Max Planck (1858–1947) recognized that mechanical momentum should be expressed relativistically as follows [105],

$$\frac{m\dot{x}}{\sqrt{1-\frac{q^2}{c^2}}}, \qquad \frac{m\dot{y}}{\sqrt{1-\frac{q^2}{c^2}}}, \qquad \frac{m\dot{z}}{\sqrt{1-\frac{q^2}{c^2}}},$$

where q is the particle speed.

Albert Einstein (1879–1955) discovered in 1909 that 'quantized' light has a momentum $\frac{hf}{c}$ [106]. Momentum soon became of crucial importance in the analysis of atomic and nuclear collisions [107].

Physics textbooks today point to the analogy between mechanical momentum generating impulse and kinetic energy producing work [108]. But what is still missing is a general definition of momentum in terms of impulse, in analogy with the established general definition of energy in terms of work.

However, to generalize momentum we must first examine the concept of inertia after Newton.

6. The concept of inertia separates into three

Newtonian inertia has two aspects. His first law, or the *inertial state*, is a naturally occurring stable state of rest, or of uniform motion, when no external forces act on the body, and when no spontaneous ruptures occur within the body. The conservation of the inertial state is similar to the conservation of momentum, but the latter is more general. We also recognize that the concept of an inertial framework is based on this law. Newton's second law, when applied to $\boldsymbol{F} = \frac{\mathrm{d}p}{\mathrm{d}t} = 0$, is not, in general, an example of the inertial state. For example, doing work at constant speed against resistance is not compatible with that state.

A body loses its inertial state if it is disturbed. If this occurs by collision with another macroscopic body, Newton's *force of inertia* immediately comes in to play. If the original inertial state was a velocity, the force of inertia is a 'forward' tensile force, acting by contact against the disturbing body. The momentum resources this force, and is opposite to the rate of change of its momentum. The same body also experiences a reversed differential stress, decelerating each layer of the body. When a contact force is applied to a free body at rest, it becomes loaded with 'a force of inertia'—a resisting stress, correlated with acceleration [109]. This is, of course, a real force. Fictional inertial forces require a separate study [110].

Euler introduced a third concept of inertia. In a publication of 1710 Leibniz had included a study of Descartes 'natural inertia'. Newton was also aware of it, but did not incorporate it explicitly into his mechanics [111]. Euler, between 1736–1765 distinguished between Newton's 'force of inertia' and what Euler called the 'quantity of inertia' [112]. For Euler, 'quantity of inertia' is not measured by the 'force of inertia', but by the ratio of the applied force to the resulting acceleration. Some of Euler's formulations of 'quantity of inertia' are almost identical with Descartes' 'natural inertia' [113].

At first, Euler argued that quantity of inertia is proportional to quantity of matter, but soon he identifies 'quantity of matter' and 'quantity of inertia' [114]. Heinrich Hertz (1857–1894) recognized some perplexity in the nineteenth century about these various concepts of inertia [115].

Max Abraham (1875–1922) in 1902 and Hendrik Lorentz (1853–1928) in 1904, went further, and demonstrated that Euler's 'quantity of inertia' (or controlling inertia) is a tensor [116]. Newton's 'inertial force', is, of course, a vector. 'Controlling inertia' adjusts the acceleration—even of a microscopic charge in an electric field. However, the 'force of inertia' does not seem to apply on that scale, and so seems less fundamental [117]. In today's physics, controlling inertia is the most common meaning of 'inertia'.

The result of this long development is that inertia has now three aspects: 'the law of inertia', the 'force of inertia' and Cartesian or controlling inertia.

7. Upgrading the definition of momentum

We are now in a position to attempt a general definition of momentum. The Lagrangian definition of momentum was, of course, applied to classical mechanics. But does it apply to special relativity, to the various forms of macroscopic electromagnetic momentum, and to the photon?

The momentum of a body will be defined as the capability to generate an impulse as it stops. Radiation impulses act during absorption.

The impulse $\boldsymbol{J}$ of the moving body on the target will be, applying the relativistic form of Newton's second law [118], and $\boldsymbol{F}$ is the forward force of inertia:

$$\begin{aligned} \boldsymbol{J} &= \int_0^t \boldsymbol{F}\,\mathrm{d}t = -\int_0^t \gamma^3 m\boldsymbol{a}\,\mathrm{d}t \\ &= -\int_v^0 \gamma^3 m\,\mathrm{d}\boldsymbol{v} \\ &= -\int_v^0 d(\gamma m\boldsymbol{v}) \\ &= \gamma m\boldsymbol{v}. \end{aligned}$$

Ideally, the target body should be infinite in mass and, therefore, after impact it, and the target, should move with an infinitesimal velocity. Alternatively, the test body should encounter an inelastic target body, with an equal and opposite momentum. In principle, therefore, relativistic momentum may be defined experimentally, rather than analytically.

It is easy to show that Lagrange's definition applies rigorously to the various forms of electromagnetic momentum as follows:

- If a finite electromagnetic wave pulse is absorbed by a static plate, then the momentum absorbed, during a short time Δt, generates an impulse as follows:

$$\boldsymbol{F}\Delta t = \Delta\boldsymbol{J} = -\frac{\partial\left[\int \frac{1}{c^2\mu_0}\boldsymbol{B}\cdot\boldsymbol{E}\,\mathrm{d}V\right]}{\partial t}\Delta t = -\frac{\partial \boldsymbol{P}_{\mathrm{EM}}}{\partial t}\Delta t = -\Delta\boldsymbol{P}_{\mathrm{EM}}.$$

  Therefore [119],

$$\boldsymbol{J} = \int_0^t \mathrm{d}\boldsymbol{J} = -\int_{\boldsymbol{P}_{\mathrm{EM}}}^0 \mathrm{d}\boldsymbol{P}_{\mathrm{EM}} = \boldsymbol{P}_{\mathrm{EM}}$$

  and the total field momentum prior to absorption is equal to the impulse generated during absorption. $\boldsymbol{F}$ may be described as the radiative force of inertia.
- Suppose a macroscopic charge is placed at rest, and in the field of a circuit or magnet. If the magnet is whisked away, or the circuit is quickly removed, or the current is reduced quickly to zero, then an impulse will act on the charge as follows,

$$\boldsymbol{J} = \int_0^t q\boldsymbol{E}\,\mathrm{d}t = -\int_0^t q\frac{\partial \boldsymbol{A}}{\partial t}\,\mathrm{d}t = -q\int_{\boldsymbol{A}}^0 \mathrm{d}\boldsymbol{A} = q\boldsymbol{A}$$

  where $\boldsymbol{A}$ is the vector potential. The capability of the charge to generate an impulse, before the current is removed, is equal to $q\boldsymbol{A}$. This is an interaction momentum, rather than a kinetic momentum [120]. In this case the vector potential, rather than the velocity, is reduced to zero during the impulse.

- If a gas atom or molecule absorbs a photon, the photon delivers a quantum mechanical impulse. The momentum transferred to the particle is $\frac{h}{\lambda}$, where h is Planck's constant, and λ is the wavelength of the photon.

In vector notation [121]

$$\boldsymbol{p} = h\boldsymbol{k} = \boldsymbol{J}$$

where $\boldsymbol{k}$ is a vector in the direction of the photon, with its magnitude equal to the number of cycles per metre. For the latter two impulses, there appears to be nothing analogous to Newton's forward force of mechanical inertia.

Momentum, as the capability to generate an impulse while it reduces to zero, includes all expressions of momentum. But what is the physical status of momentum prior to the moment of impulse?

8. Momentum as a relative 'capability' to deliver a 'finite force'

Special relativity emphasizes that momentum, and kinetic energy, are relational rather than absolute properties. Also, according to Lagrange and Rankine, momentum and kinetic energy are capabilities and capacities, respectively. Momentum is a directed capability, while energy is a scalar capacity.

During an inertial impulse a body transforms its reducing momentum into a forward force, and $-\frac{\mathrm{d}p}{\mathrm{d}t} = \boldsymbol{F}$. This means that momentum and kinetic energy can be measured operationally only while they are 'consumed'—as Leibniz puts it, that is, as $\int \boldsymbol{F}\,\mathrm{d}t$ and $\int \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{s}$, respectively. Indeed, the implicit needs to be defined in terms of the explicit. However, momentum can be expressed mathematically as a function of explicit properties of the uniformly moving body as $\gamma m \boldsymbol{\nu}$. A similar analysis applies to all kinds of momentum. Momentum, therefore (like kinetic energy), is not an explicit property (velocity), nor an analytical device (velocity potential). Indeed, physics is rich in such implicit 'capabilities', 'potentialities' and 'latent properties'.

Momentum is more versatile than impulse, because a given momentum controls a range of values of $\boldsymbol{F}$ and t so as to conserve the same equivalent impulse $\boldsymbol{J} = \boldsymbol{F}xt$. Also, the original momentum conserves the resulting momentum. Similarly, energy is more versatile than the work equivalent to it.

Although momentum is a propensity rather than a force, it is 'force-like', since it is a capability to deliver a 'finite force'. Also, momentum resources and directs all forms of collisional impulse, while energy resources work.

This paper has attempted to complete the definitional work begun by Lagrange and others on the concept of momentum. To conclude, I will examine an important tradition which challenged Newton's second law, and the concepts related to it.

9. Is Newton's second law a convention?

Newton was uncertain about the source of gravity, and states that his concept of gravity is 'purely mathematical, for I am not now considering the physical causes and sites for forces' [122]. In particular, he was unable to measure the cause or source 'accelerative force', and was always obliged to measure it indirectly, through its acceleration[2]. Newton's uncertainty about gravity created a long controversy about the definition of force in physics, which also spread into philosophy, and continues to this day [123].

[2] The measurement of the gravitational field in terms of its source was achieved by the end of the nineteenth century: [53] pp 136–8, 160–2.

The Irish philosopher, George Berkeley (1685–1753), from 1721 drew together Newton's 'purely mathematical' interpretation of gravity, and Leibniz's assertion that 'living force' cannot be known directly. Berkeley launched a powerful reductive tradition which claimed that, not only gravity, but all forces are 'occult', 'explain nothing', are 'only a mathematical hypothesis', and are 'convenient for purposes of reasoning and computation' [124].

Berkeley's theory of force attracted some physicists, especially from the late nineteenth century. Some maintained that forces can only be known through their accelerations, and others that forces are auxiliary concepts and metaphysical [125]. This reached its highest point in the early twentieth century [126]. This is an odd tradition since a common model of force among physicists was the action of a compressed or stretched spring on the target body [127]. This was clear and unhypothetical [128], and it is difficult to understand physicists who treated force as metaphysical.

Again, it is not correct that we can know a force only by its effects [129]. In fact, the causal side of force is much studied in engineering—which measures the conditions of the exploding gas in an internal combustion engine, or the thrusting water in a hydroelectric turbine. The compression in a spring—a property of the 'cause'—allows us to measure the force, as does the velocity and density of a wind.

In the mid-nineteenth century physics introduced its own related controversy about force. Newton and Euler had previously introduced force generally as a 'pressure' that acts on a static body, or on a moving body [130]. However, Lagrange accepts Newton's more restricted concept of 'motive force' as a new general concept of force:

> One understands, in general, by the term 'force' or power that which impresses, or tends to impress, movement on a body . . . .

In his gloss, by 'tends to impress movement' he means the action of forces in equilibrium. A 'tendency to movement' does, of course, exist for static forces, but they are much more than that. The flying buttresses supporting Notre Dame are in a state of sustained pressure. They are truly acting. Lagrange's definition gives only a grudging recognition to static forces.

By Lagrange's day, Newton's 'accelerative force' multiplied by the target mass had become the standard way of defining 'motive force' in advanced physics, for both gravitational forces and conventional forces. Lagrange, therefore, measures all 'accelerative forces' indirectly, by their accelerations [131]. Lagrange gives no recognition to the direct measure of force, for example, as measured in terms of calibrated springs. However, the pre-Lagrangian definition of force survives [132]:

> Anything which is capable of extending a spring is called a force.

Carl Friedrich Gauss (1777–1855) in 1833 introduced an absolute indirect measure of unit accelerative force, as that 'accelerative force which in unit time causes in [unit mass] a unit variation in velocity' [133]. With a different interpretation, this remains the absolute measure of unit force.

In the mid-nineteenth century 'accelerative force' was abandoned and replaced by 'acceleration' [134]. Ernst Mach (1838–1916), in 1868, described the 'old definition of force as the product of mass and acceleration'. But the 'old definition' was, in fact, 'mass times accelerative force', involving a shift by Mach from cause to effect. It is clear that Mach literally defines force generally as 'mass times acceleration', since he now describes Newton's second law as a 'mere identity' [135]. This is a huge category shift in the meaning of the word force, and of Newton's second law.

A significant number of physicists, especially in the early twentieth century, accepted Mach's definition of force, and thereby generated a controversy almost as intense as the earlier *vis viva* controversy—also concerned with force! [136]. In 1957, when Max Jammer wrote his *Concepts of Force*, the controversy was still apparently intense [137]. However, textbooks since the 1930s show only faint echoes of this controversy. Nevertheless, I believe that it remains lodged in the consciousness of many physicists, and needs to be examined [138].

If Mach's definition of force is taken seriously then it implies the following:

- If force *is* mass times acceleration, what causes the acceleration?
- Again, if force *is* mass times acceleration, and the acceleration is zero, does this mean that there is no force? Static forces appear to have no coherent place in Mach's definition of force. In fact, Newton's second law applies to resultant forces, only, and not necessarily to individual forces. Indeed, Newton never claimed that a 'change of motion' describes all forces.
- According to Mach, force is not an action of the source body on the target body—it is a quantity entirely located within the target body!

Mach's incoherence was the result of a series of small but confusing shifts, over more than two centuries, in the meaning of the term 'force' in mathematical physics. It was also motivated by a laudable but overdone attempt to reduce the auxiliary, the hypothetical and the metaphysical, from the concepts of physics.

A similar incoherence arises when force is *identified* as the rate of change of momentum [139]:

- If force *is* the rate of change of momentum, what generates the rate of change of momentum?
- A physical force can act when the momentum is steadily zero (in a barometer), and also when the momentum is steady (in terminal velocity).
- Traditional physical force is an action on the target body, but the rate of change of momentum is a process going on within the target body itself. These are entirely different categories. This means that, while $\boldsymbol{F}$ and $\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t}$ are quantitatively equal in certain circumstances, they are not qualitatively equal.

Mach and others describe Newton's second law, not as a law but as a 'mere identity', 'convention' or 'definition' [140]. Let us suppose that this means 'force equals mass times acceleration' is an identity in the sense that the term on the left is shorthand for the term on the right [141]. However,

- the two terms cannot be identical since, as we saw, they are qualitatively different;
- when Newton's second law is expressed in its more general form, $\sum \boldsymbol{F}_n = m\boldsymbol{a}$ (or $\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t}$), the law cannot be an identity, since the individual terms on the left are arbitrary;
- physics already knew that force is *not* equal to 'mass times acceleration' by 1916, the year when Mach died. In fact, $\boldsymbol{F} = \gamma^3 m\boldsymbol{a}$, when the $\boldsymbol{F}$ and $\boldsymbol{v}$ are parallel, and $\boldsymbol{F} = \gamma m\boldsymbol{a}$ when $\boldsymbol{F}$ and $\boldsymbol{v}$ are transverse. This does not trouble indirect force measurement at low speeds, but it entirely falsifies the claim that mass times acceleration is *identical* with force.

This emphasises that the laws of physics, and in particular $\boldsymbol{F} = \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t}$, are not physical or mathematical identities. They represent relationships between qualitatively different quantities, they are proportions or 'analogies' [142] and, allowing for their high accuracy, history suggests that it would be unwise to see them as absolute. In the last analysis, such laws are truly experimental *relationships* [143]. Furthermore, the 'quantities' of physics are bursting with qualitative meanings.

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References

[1] Philoponus J 1981 *Dictionary of Scientific Biography* vol 7 (New York: Charles Scribner's Sons) pp 134–9
Sorabji R (ed) 1987 *Philoponus and the Rejection of Aristotelian Science* (Ithaca: Cornell University Press) chapter 5
[2] Sambursky S 1987 *The Physical World of the Greeks* (London: Routledge and Kegan Paul) pp 95, 98, 99
Clagett M 1959 *The Science of Mechanics in the Middle Ages* (Madison, WI: University of Wisconsin Press) pp 505–8
[3] Clagett M 1959 *The Science of Mechanics in the Middle Ages* (Madison, WI: University of Wisconsin Press) pp 508–9
[4] Clagett M 1959 *The Science of Mechanics in the Middle Ages* (Madison, WI: University of Wisconsin Press) pp 509–10
[5] Wolff M 1987 Philoponus and the rise of preclassical mechanics *Philoponus and the Rejection of Aristotelian Science* ed R Sorabji (Ithaca: Cornell University Press) p 106
[6] Sorabji R (ed) 1987 *Philoponus and the Rejection of Aristotelian Science* (Ithaca: Cornell University Press) chapter 5
[7] Grant E 1974 *Source Book of Medieval Science* (Cambridge, MA: Harvard University Press) pp 276–80
Clagett M 1959 *The Science of Mechanics in the Middle Ages* (Madison, WI: University of Wisconsin Press) pp 520–5
[8] Dugas R 1958 *Mechanics in the Seventeenth Century* (Neuchatel: Éditions du Griffon)
Westfall R S 1971 *Force in Newton's Physics* (London: Macdonald)
Galluzzi P 1979 *Momento* (Roma: Ateno and Bizzarri) (these three texts represent monuments of scholarship and insight)
[9] Galileo G 1890–1909 *Le opere* 20 vols, ed A Favaro (Florence: Edizione Nazionali)
Galileo G 1974 *Two New Sciences* ed S Drake (Madison, WI: University of Wisconsin Press) pp xxvii, xxviii (1st edn, 1638)
Clagett M 1959 *The Science of Mechanics in the Middle Ages* (Madison, WI: University of Wisconsin Press) pp 578, 667
[10] Galileo G 1974 *Two New Sciences* ed S Drake (Madison, WI: University of Wisconsin Press) pp 172, 87, 160–3, 229, 236, 249
Westfall R S 1971 *Force in Newton's Physics* (London: Macdonald) pp 11–13, 25–7
Galluzzi P 1979 *Momenta* (Roma: Ateno & Bizzarri) pp 372–87
[11] Galileo G 1974 *Two New Sciences* ed S Drake (Madison, WI: University of Wisconsin Press) pp 156, 286, 240, 295
[12] Galileo G 1974 *Two New Sciences* ed S Drake (Madison, WI: University of Wisconsin Press) p 236
[13] Galileo G 1974 *Two New Sciences* ed S Drake (Madison, WI: University of Wisconsin Press) pp 292, 240–1, 283, 287, 289, 300–1
Galluzzi P 1979 *Momenta* (Roma: Ateno & Bizzarri) ch 7
[14] Galileo G 1974 *Two New Sciences* ed S Drake (Madison, WI: University of Wisconsin Press) pp 292, 289
[15] Kepler H 1963 *Gesammelete Werke: Epitome astronomicae Copernicanae* vol 7, ed M Caspar (Munich: C H Beck) pp 87–8
Jammer M 1997 *Concepts of Mass* (New York: Dover) p 55
[16] Kepler H 1963 *Gesammelete Werke: Epitome astronomicae Copernicanae* vol 7, ed M Caspar (Munich: C H Beck) pp 88–9
Dugas R 1958 *Mechanics in the Seventeenth Century* (Neuchatel: Éditions du Griffon) p 47
Jammer M 1997 *Concepts of Mass* (New York: Dover) p 55
[17] Kepler H 1963 *Gesammelete Werke: Epitome astronomicae Copernicanae* vol 7, ed M Caspar (Munich: C H Beck) pp 88–90
Dugas R 1958 *Mechanics in the Seventeenth Century* (Neuchatel: Éditions du Griffon) p 8
[18] Galileo G 1953 *Dialogue Concerning the Two Chief World Systems: Ptolemaic and Copernican* ed S Drake (Berkeley, CA: University of California Press) p 171 (1st edn, 1632)
Galileo G 1974 *Two New Sciences* ed S Drake (Madison, WI: University of Wisconsin Press) pp 172, 297
[19] Galileo G 1974 *Two New Sciences* ed S Drake (Madison, WI: University of Wisconsin Press) pp 286, 289, 229, 217, 230, 240
For exceptions see Westfall R S 1971 *Force in Newton's Physics* (London: Macdonald) pp 104–5

[20] Descartes R 1983 *Principles of Philosophy* ed V R Miller and R P Miller (Dordrecht: Reidel) pp 50–3, 61, 63–4, 51, 74, 183
Westfall R S 1971 *Force in Newton's Physics* (London: Macdonald) p 82
[21] Descartes R 1983 *Principles of Philosophy* ed V R Miller and R P Miller (Dordrecht: Reidel) pp 57–9, 61–8
[22] Descartes R 1983 *Principles of Philosophy* ed V R Miller and R P Miller (Dordrecht: Reidel) pp 58
Dugas R 1958 *Mechanics in the Seventeenth Century* (Neuchatel: Éditions du Griffon) p 179
[23] Cottingham J, Stoolhoff R and Murdoch D 1995 *The Philosophical writings of Descartes* vol 3 (Cambridge: Cambridge University Press) p 135
[24] Descartes R 1983 *Principles of Philosophy* ed V R Miller and R P Miller (Dordrecht: Reidel) pp 40, 58
Westfall R S 1971 *Force in Newton's Physics* (London: Macdonald) p 70
[25] Descartes R 1898, 1903 *Oeuvres* vol 2 (12 vols), ed C Adam and P Tannery (Paris: Léopold Cerf) pp 467, 543–4, 626
Descartes R 1898, 1903 *Oeuvres* vol 5, ed C Adam and P Tannery (Paris: Léopold Cerf) pp 135–4 12
Dugas R 1958 *Mechanics in the Seventeenth Century* (Neuchatel: Éditions du Griffon) pp 155–6
[26] Descartes R 1983 *Principles of Philosophy* ed V R Miller and R P Miller (Dordrecht: Reidel) pp 63–4, 51, 61, 164–5
Westfall R S 1971 *Force in Newton's Physics* (London: Macdonald) p 61
[27] Descartes R 1983 *Principles of Philosophy* ed V R Miller and R P Miller (Dordrecht: Reidel) pp 62, 64–9
Dugas R 1958 *Mechanics in the Seventeenth Century* (Neuchatel: Éditions du Griffon) pp 179–81, 412–3
[28] Dugas R 1958 *Mechanics in the Seventeenth Century* (Neuchatel: Éditions du Griffon) pp 283–94
Westfall R S 1971 *Force in Newton's Physics* (London: Macdonald) pp 291–2
[29] Wallis J 1668 The general laws of motion *Phil. Trans.* **3** 864–6
[30] Wren C 1668 The law of nature concerning the collisions of bodies *Phil. Trans.* **3** 867–8
Westfall R S 1971 *Force in Newton's Physics* (London: Macdonald) pp 203–6
[31] Huygens C 1888–1967 *Oeuvres* vol 6 (La Haye: Martinus Nijhoff) pp 334–43, 429–33
Huygens C 1888–1967 *Oeuvres* vol 16 (La Haye: Martinus Nijhoff) pp 179–81
Dugas R 1958 *Mechanics in the Seventeenth Century* (Neuchatel: Éditions du Griffon) pp 283–94
Westfall R S 1971 *Force in Newton's Physics* (London: Macdonald) pp 146–58
[32] Huygens C 1888–1967 *Oeuvres* vol 16 (La Haye: Martinus Nijhoff) 179–81 (circa 1659)
Westfall R S 1971 *Force in Newton's Physics* (London: Macdonald) pp 156, 190
[33] Wallis J 1670–1671 *Mechanica sive de Motu* (London: Mosis Pitt) pp 686–98
Dugas R 1958 *Mechanics in the Seventeenth Century* (Neuchatel: Éditions du Griffon) pp 280–89
[34] Wallis J 1668 The general laws of motion *Phil. Trans.* **3** 864–6
Wallis J 1670/71 *Mechanica sive de Motu* (London: Mosis Pitt) pp 660–2
[35] Wallis J 1670 *Mechanica sive de Motu* (London: Mosis Pitt) p 94
Westfall R S 1971 *Force in Newton's Physics* (London: Macdonald) p 235
[36] For a penetrating study see Warren J W 1979 *Understanding Force* (London: John Murray)
[37] Westfall R S 1971 *Force in Newton's Physics* (London: Macdonald) pp 9, 231–2
[38] Galileo G 1974 *Two New Sciences* ed S Drake (Madison, WI: University of Wisconsin Press) pp 11–146
[39] Huygens C 1888–1967 *Oeuvres* vol 16 (La Haye: Martinus Nijhoff) p 263
Westfall R S 1971 *Force in Newton's Physics* (London: Macdonald) pp 168–9, 214
de Breteuil Châtelet E 1740 *Institutions de physique* (Paris: Prault) p 399
Westfall R S 1971 *Force in Newton's Physics* (London: Macdonald) pp 168–9, 214
[40] Hooke R 1678 *Lectures de Potentia Restitutiva or of Spring* (London: Jan Martyn) pp 1–6
[41] Aristotle 1995 *Mechanics* ed J Barnes (Princeton: Princeton University Press) pp 1299–319
Singer C, Holmyard E J, Hall A R and Williams T I 1954, 1967–1979 *A History of Technology* vol 2 (6 vols) (Oxford: Clarendon) chapter 18 (Machines)
[42] Aristotle 1995 *Physics* ed J Barnes (Princeton, NJ: Princeton University Press) pp 417–8
[43] Westfall R S 1971 *Force in Newton's Physics* (London: Macdonald) pp 232–3
[44] Newton I 1999 *The Principia* ed I B Cohen and A Whitman (Berkeley, CA: University of California Press) pp 417–420 (1st edn, 1687)
Herivel J 1965 *The Background to Newton's Principia* (Oxford: Clarendon) p 138
[45] Newton I 1999 *The Principia* ed I B Cohen and A Whitman (Berkeley, CA: University of California Press) p 405, 687–8, 762, def 4
[46] Newton I 1999 *The Principia* ed I B Cohen and A Whitman (Berkeley, CA: University of California Press) pp 429–30, 417–8, corr. 1
[47] Newton I 1999 *The Principia* ed I B Cohen and A Whitman (Berkeley, CA: University of California Press) pp 424, 417
[48] Newton I 1999 *The Principia* ed I B Cohen and A Whitman (Berkeley, CA: University of California Press) pp 406–8, def. 6–8
[49] Newton I 1999 *The Principia* ed I B Cohen and A Whitman (Berkeley, CA: University of California Press) pp 404–8
[50] Newton I 1999 *The Principia* ed I B Cohen and A Whitman (Berkeley, CA: University of California Press) pp 405, 417
Westfall R S 1971 *Force in Newton's Physics* (London: Macdonald) pp 377, 387, 450–2, 454–6

[51] Newton I 1999 *The Principia* ed I B Cohen and A Whitman (Berkeley, CA: University of California Press) pp 404–5, 416, def. 3,4
Wood J 1824 *The Principles of Mechanics* (Cambridge: Cambridge University Press) p 8
Herivel J 1965 *The Background to Newton's Principia* (Oxford: Clarendon) p 231
Westfall R S 1971 *Force in Newton's Physics* (London: Macdonald) pp 346, 490
[52] Newton I 1999 *The Principia* ed I B Cohen and A Whitman (Berkeley, CA: University of California Press) pp 407, 416–7, 438, 454, 562, 767–8
Westfall R S 1971 *Force in Newton's Physics* (London: Macdonald) pp 424, 496
[53] Roche J 1998 *The Mathematics of Measurement: a Critical History* (London: Athlone/Springer) p 158
[54] Newton I 1999 *The Principia* ed I B Cohen and A Whitman (Berkeley, CA: University of California Press) pp 407, 430
[55] Newton I 1999 *The Principia* ed I B Cohen and A Whitman (Berkeley, CA: University of California Press) pp 407, 424, 552, 700–1
Westfall R S 1971 *Force in Newton's Physics* (London: Macdonald) pp 344–5, 472
[56] Atwood G 1784 *Treatise on the Rectilinear Motion and Rotation of Bodies* (Cambridge: Cambridge University Press) pp 109–12, 291–337
DeLauncey C 1862 *Cours Élementaire de Mechanique* (Paris: Victor Massin) p 117
[57] Newton I 1999 *The Principia* ed I B Cohen and A Whitman (Berkeley, CA: University of California Press) pp 407, 414, 713, 715, 719, 768, 771–2
[58] Newton I 1999 *The Principia* ed I B Cohen and A Whitman (Berkeley, CA: University of California Press) pp 407, 437–8, 454, 462–3, 767–8
[59] Newton I 1999 *The Principia* ed I B Cohen and A Whitman (Berkeley: University of California Press) pp 459–61, 529–33, 803–5
[60] Newton I 1999 *The Principia* ed I B Cohen and A Whitman (Berkeley, CA: University of California Press) pp 404–5, def. 3
[61] Newton I 1999 *The Principia* ed I B Cohen and A Whitman (Berkeley, CA: University of California Press) pp 404–5, def. 3
Newton I 1999 *The Principia* ed I B Cohen and A Whitman (Berkeley, CA: University of California Press) p 417, law 3
[62] Newton I 1999 *The Principia* ed I B Cohen and A Whitman (Berkeley, CA: University of California Press) p 427
[63] Newton I 1999 *The Principia* ed I B Cohen and A Whitman (Berkeley, CA: University of California Press) pp 417, 420–28
[64] Dugas R 1958 *Mechanics in the Seventeenth Century* (Neuchatel: Éditions du Griffon) pp 466–7, 474, 412
[65] Maxwell J C 1872 *Theory of Heat* (London: Longmans Green) p 88
[66] Newton I 1999 *The Principia* ed I B Cohen and A Whitman (Berkeley, CA: University of California Press) pp 405–8
[67] Newton I 1999 *The Principia* ed I B Cohen and A Whitman (Berkeley, CA: University of California Press) pp 462–3, 794–6, 802–5
[68] Newton I 1999 *The Principia* ed I B Cohen and A Whitman (Berkeley, CA: University of California Press) p 407
[69] Newton I 1999 *The Principia* ed I B Cohen and A Whitman (Berkeley, CA: University of California Press) p 407
[70] Newton I 1999 *The Principia* ed I B Cohen and A Whitman (Berkeley, CA: University of California Press) p 407
[71] Newton I 1999 *The Principia* ed I B Cohen and A Whitman (Berkeley, CA: University of California Press) pp 423, 772
[72] Maxwell J J C 1865 A dynamical theory of the electromagnetic field *Phil. Trans.* **155** 459–512, 493
[73] Desaguliers J 1734 *A Course of Experimental Philosophy* (London: John Senex) p 43
[74] Leibniz G 1956 *Philosophical Letters and Letters* vol 1 (2 vols), ed L E Loemker (Chicago, IL: University of Chicago Press) p 455
[75] Leibniz G 1956 *Philosophical Letters and Letters* vol 1 (2 vols), ed L E Loemker (Chicago, IL: University of Chicago Press) pp 456–8, 461–2
Dugas R 1958 *Mechanics in the Seventeenth Century* (Neuchatel: Éditions du Griffon) p 475
Desaguliers J 1734 *A Course of Experimental Philosophy* (London: John Senex) p 2
Westfall R S 1971 *Force in Newton's Physics* (London: Macdonald) p 285
[76] Leibniz G 1956 *Philosophical Letters and Letters* vol 1 (2 vols), ed L E Loemker (Chicago, IL: University of Chicago Press) 456–7, 462
Leibniz G 1956 *Philosophical Letters and Letters* vol 2 (2 vols), ed L E Loemker (Chicago, IL: University of Chicago Press) p 716
Dugas R 1958 *Mechanics in the Seventeenth Century* (Neuchatel: Éditions du Griffon) p 475
Clarke S 1727 The proportion of velocity and force in bodies in motion *Phil. Trans.* **35** 381–8, 382–3
de Breteuil Châtelet E 1740 *Institutions de physique* (Paris: Prault) p 398
d'Alembert J 1796 *Traité de dynamique* (Paris: Fuchs) p xviii (1st edn, 1743)
Costabel P 1973 *Leibniz and dynamics* ed R Maddison (Paris: Hermann) p 48

Westfall R S 1971 *Force in Newton's Physics* (London: Macdonald) pp 285–7
Iltis C 1971 Leibniz and *viv viva* controversy *ISIS* **62** 21–35, 25
[77] Leibniz G 1956 *Philosophical Letters and Letters* vol 1 (2 vols), ed L E Loemker (Chicago, IL: University of Chicago Press) pp 458–6
Clarke S 1727 The proportion of velocity and force in bodies in motion *Phil. Trans.* **35** 381–8, 382
Dugas R 1958 *Mechanics in the Seventeenth Century* (Neuchatel: Éditions du Griffon) pp 466–473, 482
Westfall R S 1971 *Force in Newton's Physics* (London: Macdonald) p 285
Iltis C 1971 Leibniz and *viv viva* controversy *ISIS* **62** 21–35, 25
[78] Leibniz G 1956 *Philosophical Letters and Letters* vol 2 (2 vols), ed L E Loemker (Chicago, IL: University of Chicago Press) pp 730–1
Dugas R 1958 *Mechanics in the Seventeenth Century* (Neuchatel: Éditions du Griffon) pp 475–6, 479
[79] Leibniz G 1956 *Philosophical Letters and Letters* vol 2 (2 vols), ed L E Loemker (Chicago, IL: University of Chicago Press) p 717
[80] Leibniz G 1956 *Philosophical Letters and Letters* vol 1 (2 vols), ed L E Loemker (Chicago, IL: University of Chicago Press) p 457
Leibniz G 1956 *Philosophical Letters and Letters* vol 2 (2 vols), ed L E Loemker (Chicago, IL: University of Chicago Press) p 715
Dugas R 1958 *Mechanics in the Seventeenth Century* (Neuchatel: Éditions du Griffon) p 472
[81] Leibniz G 1956 *Philosophical Letters and Letters* vol 2 (2 vols), ed L E Loemker (Chicago, IL: University of Chicago Press) p 719
Dugas R 1958 *Mechanics in the Seventeenth Century* (Neuchatel: Éditions du Griffon) p 474
[82] Clarke S 1727 The proportion of velocity and force in bodies in motion *Phil. Trans.* **35** 381–8, 383
Desaguliers J 1722 An account of some experiments made to prove that the force of moving bodies is proportional to their velocities *Phil. Trans.* **32** 269–79
Camus C 1738 Sur l'action d'une ballet de mousquet *Mem. Acad. R. Sci.* **Anneé 1738** 147–58
Dugas R 1958 *Mechanics in the Seventeenth Century* (Neuchatel: Éditions du Griffon) p 472
Westfall R S 1971 *Force in Newton's Physics* (London: Macdonald) p 300
[83] Borelli A 1686 *De vi percussionis* (Leyden: Petrum va Or) p 120 (1st edn, 1667)
MacLaurin C 1724 Les loix du choc des corps *Pieces qui remporté le la prix de l'Academie Royale des Sciences* (Paris: Joubert) pp 9, 10
d'Alembert J 1796 *Traité de dynamique* (Paris: Fuchs) p xiii (1st edn, 1743)
[84] Hankins T 1965 Eighteenth-century attempts to resolve the vis-viva controversy *ISIS* **56** 280–97, 286
[85] d'Alembert J 1796 *Traité de dynamique* (Paris: Fuchs) pp viii, ix, xv–xviii, xix, xxi (1st edn, 1743)
Euler L 1745 De la force de percussion *Hist. Acad. R. Sci. Belles Lett. Berl.* **Anneé 1745** 22, 24, 27–8
[86] d'Alembert J 1796 *Traité de dynamique* (Paris: Fuchs) pp iii, ix, 212–3 (1st edn, 1743)
Euler L 1745 De la force de percussion *Hist. Acad. R. Sci. Belles Lett. Berl.* **1745** 28, 30, 34
[87] d'Alembert J 1796 *Traité de dynamique* (Paris: Fuchs) p xxi (1st edn, 1743)
Euler L 1745 De la force de percussion *Hist. Acad. R. Sci. Belles Lett. Berl.* **1745** 21–53, 28, 31–4, 37–53
[88] d'Alembert J 1796 *Traité de dynamique* (Paris: Fuchs) pp xviii, xix (1st edn, 1743)
Euler L 1745 De la force de percussion *Hist. Acad. R. Sci. Belles Lett. Berl.* **1745** 22–7
[89] Varignon P 1700 Mémoire générale de déterminer les forces, les vitesses, les espaces et les temps *Mem. Acad. R. Sci.* **1700** 22–7, 27
Varignon P 1719 Comparison des vitesses des corps *Mem. Acad. R. Sci.* **1719** 195–229, 198–200
d'Alembert J 1796 *Traité de dynamique* (Paris: Fuchs) pp xviii, 26, 253 (1st edn, 1743)
Euler L 1745 De la force de percussion *Hist. Acad. R. Sci. Belles Lett. Berl.* **1745** 33–5, 51–2
Dugas R 1958 *Mechanics in the Seventeenth Century* (Neuchatel: Éditions du Griffon) pp 467, 471
[90] Camus C 1738 Sur l'action d'une ballet de mousquet *Mem. Acad. R. Sci.* **1738** 147–58
d'Alembert J 1796 *Traité de dynamique* (Paris: Fuchs) p xix (1st edn, 1743)
Euler L 1745 De la force de percussion *Hist. Acad. R. Sci. Belles Lett. Berl.* **1745** 50–2
[91] Morin A 1855 *Méchaniques* (Paris: Hachette) p 63
[92] Lagrange J L 1853 *Méchanique analytique* (Paris: Courcier) p 214 (1st edn, 1788)
[93] Lagrange J L 1788 *Méchanique analytique* (Paris: De Saint) pp 160, 166–7
[94] Lagrange J L 1788 *Méchanique analytique* (Paris: De Saint) p 168
[95] Lagrange J L 1788 *Méchanique analytique* (Paris: De Saint) p 160
[96] Prony R 1810,5 *Leçons de méchanique analytique* vol 2 (2 vols) (Paris: École des points et chauss*ées*) pp 82–9
Poisson S 1833 *Traité de mécanique* vol 1 (2 vols) (Paris: Bachelier) pp 232–3
[97] Belanger J B 1856 *Cours de méchanique et machines* vol 1 (Paris: École Polytechique) pp 131–2
Belanger J B 1856 *Cours de méchanique et machines* vol 2 (Paris: École Polytechique) pp 4–5
[98] Belanger J B 1856 *Cours de méchanique et machines* vol 1 (Paris: École Polytechique) pp 131–2
Belanger J B 1856 *Cours de méchanique et machines* vol 2 (Paris: École Polytechique) pp 4–5
Roche J 1998 *The Mathematics of Measurement: a Critical History* (London: Athlone/Springer) p 159
[99] Maxwell J C 1872 *Theory of Heat* (London: Longmans Green) p 88
Maxwell J C 1990 *The Scientific Letters and Papers of James Clerk Maxwell* vol 1 ed P Harman (Cambridge: Cambridge University Press) p 513

Maxwell J C 1990 *The Scientific Letters and Papers of James Clerk Maxwell* vol 2 ed P Harman (Cambridge: Cambridge University Press) p 423
[100] Jude R H and Gossin H 1899 *Physics* (London: Chapman and Hall) pp 43–5
Culler J A 1906 *A Text-book of Physics* (Philadelphia, PA: Lippincott) pp 46–7
Kimball A 1912 *A College Text-book of Physics* (London: Bell) p 21
Young H and Freedman R 2004 *University Physics* (San Francisco, CA: Pearson/Addison-Wesley) p 285
[101] Rankine W 1881 *Miscellaneous Scientific Papers* (London: Charles Griffin) p 217
Roche J What is potential energy? *Eur. J. Phys.* **24** 185–96, 187
[102] Thomson W and Tait P G 1872 *A Treatise of Natural Philosophy* (part 1) (Cambridge: Cambridge University Press) p 53
Roche J What is potential energy? *Eur. J. Phys.* **24** 185–96, 192
[103] Expressed by $\frac{EH}{4\pi}$: Poynting J 1884 On the transfer of energy in the electromagnetic field *Phil. Trans.* **175** 343–61, 349
[104] Expressed by $\mu c \int \frac{i}{R} \frac{dx}{ds} ds$. This is equivalent to $q\boldsymbol{A}$. Thomson J J 1904 On momentum in the electric field *Phil. Mag.* **8** 331–56, 348–9
[105] Planck M 1906 Das prinzip der relativität und die grundgleichungen der mechanik *Verh. Dtsch. Phys. Ges.* **7** 136–41, 139
Whittaker E T 1961–1962 *A History of the Theories of Aether and Electricity* vol 2 (2 vols) (London: Thomas Nelson) pp 44–51
[106] Einstein A 1989 *The Collected Papers* ed A Beck (Princeton, NJ: Princeton University Press) pp 369, 392–4
[107] Pais A 1986 *Inward Bound: of Matter and Forces in the Physical World* (Oxford: Clarendon) chapters 3–6
[108] Michels W, Correll M and Patterson A 1968 *Foundations of Physics* (Princeton, NJ: Von Nostrand) pp 99–100
Young H and Freedman R 2004 *University Physics* (San Francisco, CA: Pearson/Addison-Wesley) pp 284–5
[109] Mabie H and Reinholtz C 1987 *Mechanisms and Dynamics of Machinery* (New York: Wiley) p 396
[110] Warren J W 1979 *Understanding Force* (London: John Murray) pp 13–9
[111] Leibniz G 1956 *Philosophical Letters and Letters* vol 2 (2 vols), ed L E Loemker (Chicago, IL: University of Chicago Press) pp 836, 840, 1162–7
Debus A 1972 *Science, Medicine and Society in the Renaissance* (London: Heinemann) pp 199–211
Barbour J 1989 *Absolute or Relative Motion* (Cambridge: Cambridge University Press) pp 328–39
[112] Euler L 1745 De la force de percussion *Hist. Acad. R. Sci. Belles Lett. Berl.* **1745** 22–4
Euler L 1765 Theoria Motus Corporum Solidorum seu Rigidum *Opera omnia (series secunda)* vol 3 ed P Stäckl (Leipzig: Teubner and Orell Fusli) p 73 (1912–1964, 30 vols)
[113] Euler L *et al* 1765 Theoria Motus Corporum Solidorum seu Rigidum *Opera omnia (series secunda)* vol 3, ed P Stäckl (Leipzig: Teubner and Orell Fusli) pp 70–8 (1912–1964, 30 vols) Euler does not seem to mention Cartesian inertia explicitly, but Euler's *Opera* contains 30 volumes!
[114] Euler L *et al* 1736 *Mechanica sive motus scientis analytica exposita* vol 1 (Petropolis: Academia scientiarum) pp 57, 67
Euler L 1745 De la force de percussion *Hist. Acad. R. Sci. Belles Lett. Berl.* **1745** 24
Jammer M 1997 *Concepts of Mass* (New York: Dover) pp 88–9
[115] Hertz H 1956 *The Principles of Mechanics* (New York: Dover) p 6 (1st German edn, 1894)
[116] Goldberg S 1970–1971 The Abraham theory of the electron *Arch. Hist. Exact Sci.* **7** 7–25 14, 18
[117] Lorentz H A 1935–1939 *Collected Papers* vol 5 (The Hague: Martinus Nijhoff) p 28
[118] Okun L 1989 The concept of mass *Sov. Phys.—Usp.* **32** 633–8, 632
Roche J 2005 What is mass? *Eur. J. Phys.* **26** 225–242
[119] Poynting J 1884 On the transfer of energy in the electromagnetic field *Phil. Trans.* **175** 343–61, 349
Jackson J D 1975 *Classical Electrodynamics* (New York: Wiley) p 238
Ditchburn R W 1976 *Light* (London: Academic) pp 633–68
Laudon R 2000 *The Quantum Theory of Light* (Oxford: Oxford University Press) pp 40–4
[120] Thompson J J 1904 On momentum in the electric field *Phil. Trans.* **8** 331–56, 348–9
Roche J 1990 A critical history of the vector potential *Physicists Look Back* ed J Roche (Bristol: Adam Hilger) pp 144–68
[121] Ditchburn R W 1976 *Light* (London: Academic) pp 633–68
Laudon R 2000 *The Quantum Theory of Light* (Oxford: Oxford University Press) pp 40–4
[122] Newton I 1999 *The Principia* ed I B Cohen and A Whitman (Berkeley, CA: University of California Press) pp 407, 408
Dugas R 1958 *Mechanics in the Seventeenth Century* (Neuchatel: Éditions du Griffon) pp 416–7
[123] Newton I 1999 *The Principia* ed I B Cohen and A Whitman (Berkeley, CA: University of California Press) p 392
Dugas R 1958 *Mechanics in the Seventeenth Century* (Neuchatel: Éditions du Griffon) pp 16–7, 549–61
Jammer M 1999 *Concepts of Force* (New York: Dover) chapter 11
[124] Berkeley G 1975 On Motion *Philosophical Works* (London: Dent) pp 211–4
Jammer M 1999 *Concepts of Force* (New York: Dover) p 204
[125] Lagrange J L 1853 *Méchanique analytique* (Paris: Courcier) p 209
Poincaré H 1905 *Science and Hypothesis* (London: Walter Scott) p 98
Jammer M 1999 *Concepts of Force* (New York: Dover) pp 204, 207, 208–9, 215–7

[126] Jammer M 1999 *Concepts of Force* (New York: Dover) pp 204, 207, 208–9, 215–7
[127] Wilson W P 1850 *A Treatise on Dynamics* (Cambridge: Macmillan) pp 18–19
[128] Belanger J B 1856 *Cours de méchanique et machines* vol 1 (Paris: École Polytechique) p 106
Duhamel J 1862 *Cours de Mechanique* (Paris: Mallet-Bachelier) p 92
Love A E 1897 *Theoretical Mechanics* (Cambridge: Cambridge University Press) p 1
Poincaré H 1914 *Science and Method* (London: Nelson) pp 137–42
Jammer M 1999 *Concepts of Force* (New York: Dover) p 217
[129] Whewell W 1832 *Elementary Treatise on Mechanics* (Cambridge: Deighton) p 138
Abbott A F 1978 *Ordinary Level Physics* (London: Heinemann) p 13
[130] Herivel J 1965 *The Background to Newton's Principia* (Oxford: Clarendon) p 231
Euler L 1745 De la force de percussion *Hist. Acad. R. Sci. Belles Lett. Berl.* **Anneé 1745** 24–5, 30–4
[131] Lagrange J L 1853 *Méchanique analytique* (Paris: Courcier) pp 1, 161, 167–8
[132] Fraunenfelder P and Huber P 1966 *Introduction to Physics* (Oxford: Pergamon) p 24
Also see Wilson W P 1850 *A Treatise on Dynamics* (Cambridge: Macmillan) pp 18–19
[133] Gauss C F 1834 Mesure absolue de l'intensité du magnétisme terrestre *Ann. Chim. Phys.* **57** 1–69, 13
Jamin J 1882 *Cours de Physique* (Paris: Gauthier-Villars) p 8
[134] Tait P G 1876 Force *Nature* **14** 459–63, 461
Roche J 1998 *The Mathematics of Measurement: a Critical History* (London: Athlone/Springer) pp 157–8
[135] Mach E 1911 *History and Root of the Principle of Conservation of Energy* ed P E Jourdain (Chicago, IL: Open Court) pp 84–5 (1st German edn, 1871)
Mach E 1960 *The Science of Mechanics* ed T McCormach (La Salle, IL: Open Court) pp 300–4 (1st German edn, 1883)
[136] Tait P G 1876 Force *Nature* **14** 459–63, 459
Daniell A 1888 *Text Book of Physics* (London: Macmillan) pp 17–8
Love A E 1897 *Theoretical Mechanics* (Cambridge: Cambridge University Press) p 182
Campbell N R *Physics: the Elements* (Cambridge: Cambridge University Press) pp 41–2
Crew H 1909 *General Physics* (New York: Macmillan) p 69
[137] Jammer M 1999 *Concepts of Force* (New York: Dover) chapter 12 (1st edn, 1957)
[138] Abbott A F 1978 *Ordinary Level Physics* (London: Heinemann) p 13
Nelkon M and Parker P 1984 *Advanced Level Physics* (London: Heinemann) pp 13, 18
Keller F, Gettys W and Skove M 1993 *Physics* (New York: McGraw-Hill) pp 87, 95
[139] Broad C P 1925 *Scientific Thought* (London: Kegan Paul) pp 161–2, 165
Michels W, Correll M and Patterson A *Foundations of Physics* (Princeton, NJ: Von Nostrand) p 96
Rosser W 1991 *Introductory Special Relativity* (London: Taylor and Francis) p 4 (http://physics.about.com/cs/mechanics/g/Force.htm; accessed 12.4.04)
[140] Poincaré H 1905 *Science and Hypothesis* (London: Walter Scott) pp 40
but see Warren J W 1979 *Understanding Force* (London: John Murray) p 9
[141] Campbell N R *Physics: the Elements* (Cambridge: Cambridge University Press) pp 40–41
Broad C P *Scientific Thought* (London: Kegan Paul) pp 161–2
[142] Von Helmholtz H 1882 On systems of absolute measurement for electric and magnetic quantities *Phil. Mag.* **14** 430–40, 431
[143] Williams D J 1974 *Force, Matter and Energy* (London: The English Universities Press) pp 557–9
[144] Maupassant G de 1982 *Selected Short Stories* ed R Colet (Harmondsworth: Penguin) p 10